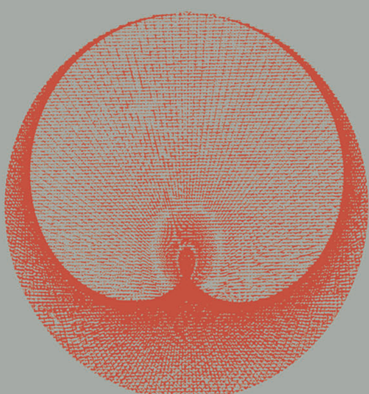
Lecture Notes in
Computer Science 1984

Joe Marks (Ed.)

Graph Drawing

8th International Symposium, GD 2000
Colonial Williamsburg, VA, USA, September 2000
Proceedings



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Volume Editor

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The Visual Representation of Information Structures

Colin Ware

Data Visualization Research Lab
Center for Coastal and Ocean Mapping
University of New Hampshire Durham, NH 03924.

Abstract. It is proposed that research into human perception can be applied in designing ways to represent structured information. This idea is illustrated with four case studies. (1) How can we design a graph so that paths can be discerned? Recent results in the perception of contours can be applied to make paths easier to perceive in directed graphs. (2) Should we be displaying graphs in 3D or 2D space? Research suggests that larger graphs can be understood if stereo and motion parallax depth cues are available. (3) How can heterogeneous information structures be best represented? Experiments show using structured 3D shape primitives make diagrams that are easier to discover and remember. (4) How can causal relationships be displayed? Michotte's work on the perception of causality suggests that causal relationships can be represented using simple animations. The general point of these examples is that data visualization can become a science based on the mapping of data structures to visual representations. Scientific methods can be applied both in the development of theory and testing the value of different representations.

1 Introduction

The human visual system provides by far the highest bandwidth channel into the brain (70% of all receptors, more than 40% of cortex devoted to vision). However, the visual machinery does not analyze all patterns equally well. This paper illustrates ways in which we may apply what is known about human perception to data representation.

2 Finding Paths in Directed Graphs

Recent work on the gestalt concept of "good continuity" provides us with a detailed description of the conditions under which continuous contours are perceived [2]. These results can be applied directly to the problem of emphasizing important paths in graphs. For example paths drawn so that edge curvature is minimized will be easier to be perceived. This is illustrated in Figure 1. In addition nodes should be placed in rough alignment and should consist of oriented elongated symbols.

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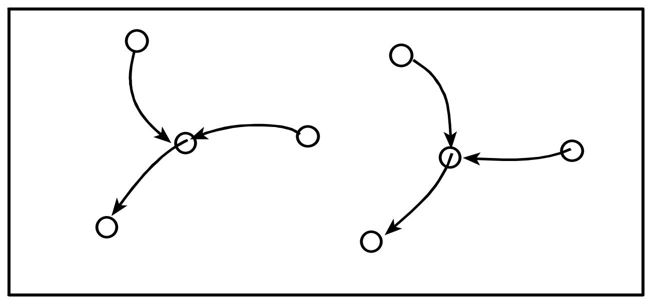

Fig. 1. Different paths are emphasized on the same graph.

3 Visualizing Graphs in 3D

There is a debate about whether information should be displayed in 3D or in 2D. A study by Ware and Franck examined the value of different depth cues in a task relating to tracing paths in graphs [8]. The results showed that over a wide range of graph sizes using stereopsis allowed about 60% more to be seen and using motion parallax allowed about 130% more to be seen. As expected, the perspective depth cue was not important for this particular task.

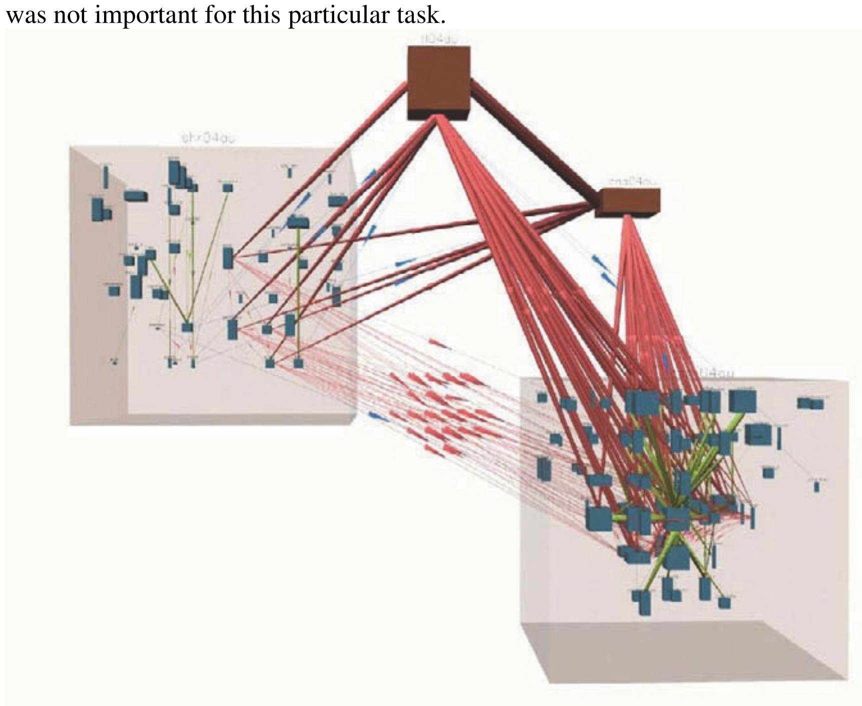

Fig. 2. This large graph represents the structure of a 6 million line program. (Image copyright Glenn Franck and Colin Ware. Reprinted with permission)

4 Using Geons to Represent Structured Information

Research by Marr [5] and by Biederman [1] suggests that complex objects are broken down by the visual system into 3d "geon" primitives. Irani and Ware have shown that diagrams constructed using these primitives are much easier to analyze and recognize in comparison with conventional box-line diagrams [3]. Figure 3. illustrates a "geon" diagram [3].

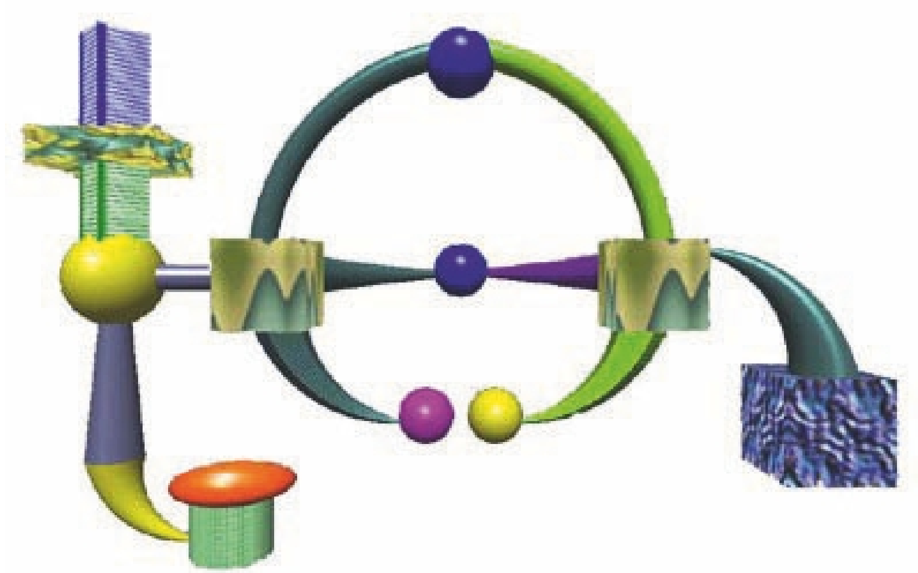

Fig. 3. Geons are 3D primitives of object perception. Diagrams constructed from geon elements may be both more accurately interpreted and recalled (image copyright Pourang Irani: reprinted with permission)

5 Simple Animation for Representing Causal Relations

The work of Michotte [4] suggests that given simple animations and the appropriate timing of events people can "directly" perceive causality in the same sense that they directly perceive a connection between two objects linked by a line. Ware et al [7] applied this phenomenon of causality perception to the representation of causal relationships in statistical graphs. One of the methods for representing causality was to have a ball move, striking a "node" in a graph apparently causing it to vibrate (Figure 4).

Fig. 4. Causality can be represented by a ball striking a node "causing" it to oscillate.

6 Conclusion

Semiotics is the study of symbols and the way in which they display meaning. With advances in the science of human perception we can lay the foundation for a science of semiotics. This science allows us to develop testable theories relating to the mapping for information structures to visual representations. The ultimate goal is to map data in ways that take full advantage of the huge processing power inherent in the human visual system [6].

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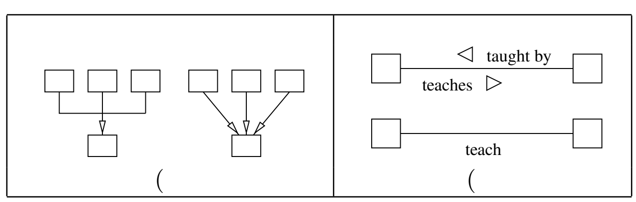

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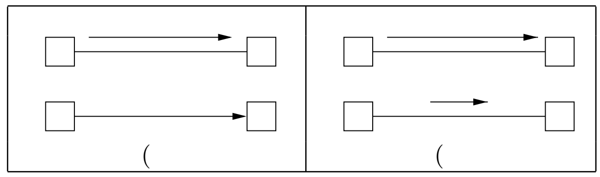

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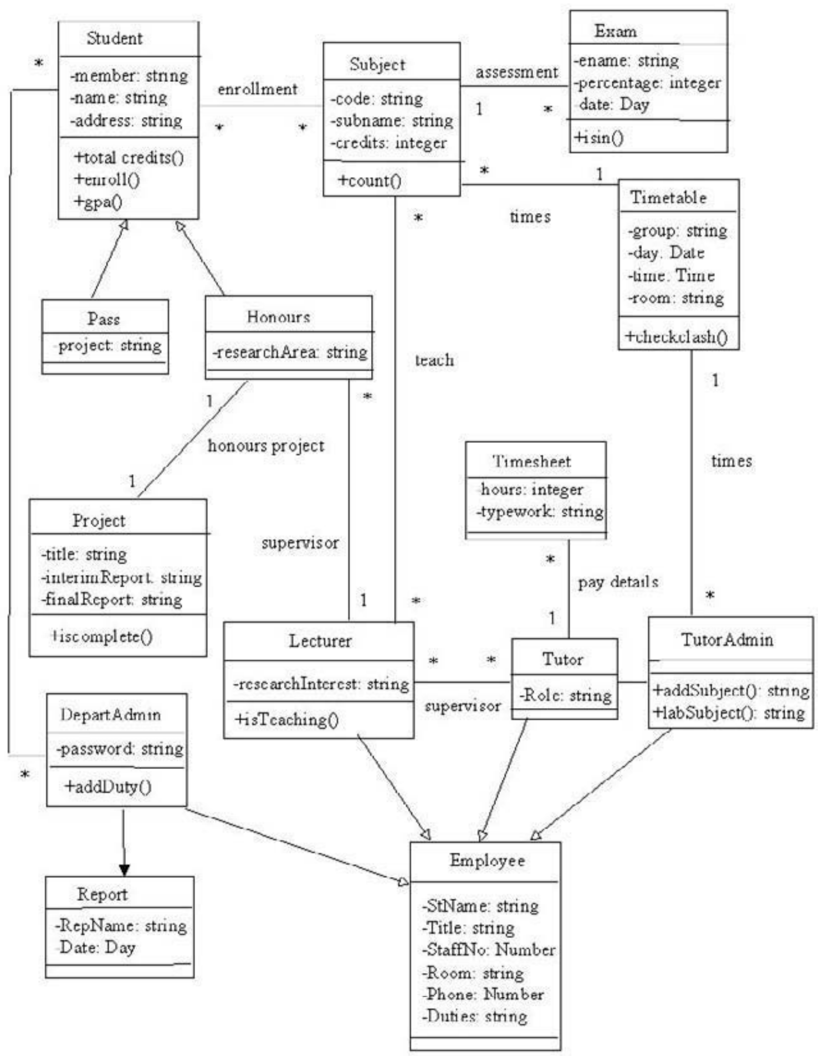

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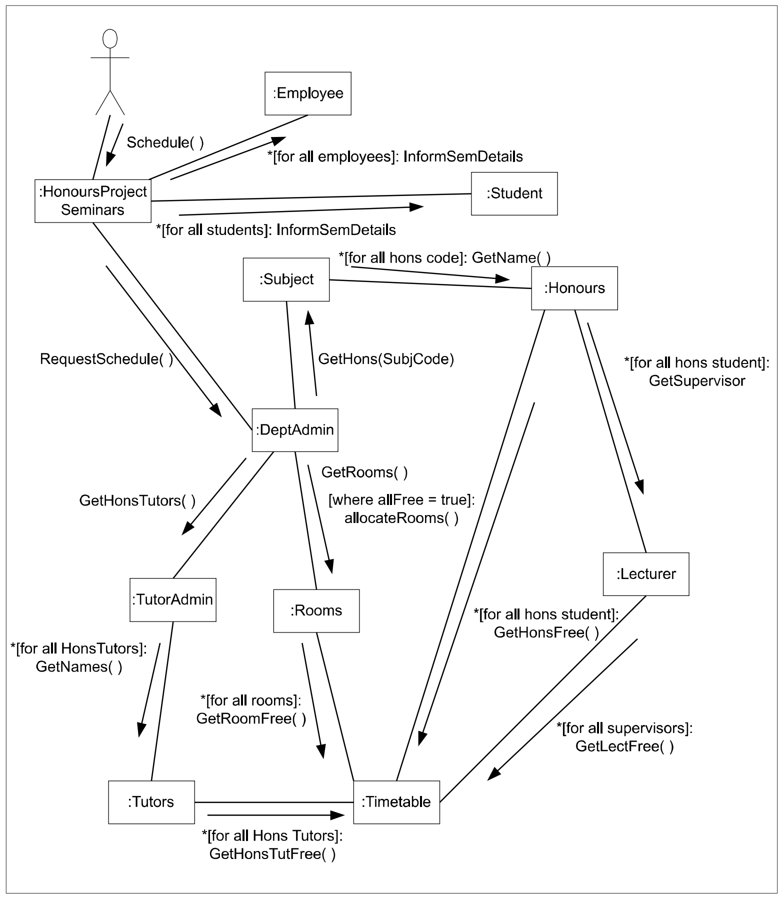

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q stio of o si il r t o r i gs of gr p s r ris s i sit tio s.
i t r tiv gr p r i g t gr p b i g vis li g s ov r ti
it is i port t to pr s rv t s r's t l p" 2 so s o s ot sp
lot of ti r l r i g t r i g ft r p t . i tio provi
s oot tr sitio b t r i gs b t is of li it s if t r i gs r
too i r t. l o t j st t ppli tio s isti g r i g is o i
to i prov st ti q lit it o t stro i g t s r's t l p.
ot r ppli tio is i i i g or bro si g l rg s ts of gr p s.
gr p lti r i g s st (i l t. l.) pro s r i gs
of gr p its gr p bro s r rr g s t r i gs so t t si il r o s r
r ot r to lp t s r vig t t s st 's r spo s s. i il riti s
b t r i gs lso b s s b sis for i i g r tri v l.
r t r riti g r og itio ritt r t r b tr sfor
i to gr p o p r to t b s of r t rs to t los st t .
t M b si il rit s r so t t s r 's v l is t
r i gs r i ti l. or r to b s f l M s o l s tisf t r prop rti s

t t iv r i gs D D $M(D, D_\theta)$ s o l v t i i
v l for t gl s r o l r port s givi g t b st t r D_θ
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* c upp t i p t t ti l ci c u ti u t -
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iv r i gs D D D $M(D,D\ ) < M(D,D\ )$ if o l if s r o l s t t D is or lik D t D is lik D.
t iv r i gs D D D $M(D,D\ )$ $\frac{\cdot}{c}M(D,D\ )$ if o l if s r o l s t t D is c ti s or lik D t D is lik D.

is p p r s rib s s r st i t to r ss rot tio or ri g to plor t o for r ssi g g it . t ot b oll t ir tl for t g it p rt b s it is v r i lt to ssig ri- l si il rit v l s to p irs of r i gs or so t j gi g or ri g (i k lgr). s r s lt ot r t (.g. r spo s ti s) pr s to b r l t to si il rit st b oll t i st . ss ptio of t s it bilit of t t b p rti ll t st b t r i i g if si g t t to k or ri g isio s b t r i gs is o sist t it t s r r spo s s. is st i prov s o o r pr vio s ork 3 i s v r l s

t **t** l rg r pool of s rs (3 i tot l) s s for t r i i g t orr t" b vior for t s r .
t s rs o l p ir is j g ts b t r i gs r t r t b i g sk to or r l rg r s t.
t **t** g it s ot r ss i 3 .
t **t** pr vio s r i g lig t t o llo o r i g to b s l rbitr ril s ll it r sp t to t ot r; t t o k ps t s s l f tor for bot r i gs.
t or t os s r s o p t it p irs of poi ts p irs i volvi g poi ts fro t s v rt r skipp .
v r l s r s v b i l .

s rib t p ri t l s t p i tio 2 t s r s v l t i tio 3 t r s lts i tio 4 o l sio s ir tio s for f t r ork i tio .

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is st fo s s o si il rit s r s for ort ogo l r i gs of rl t s gr p . rl t s gr p " s t t t gr p s i r b o l s ll b r of v rt g i s rtio s l tio s. fo s o ort ogo- l r i gs is otiv t b t v il bilit of ort ogo l r i g lgorit p bl of pro i g r i gs of t s gr p b t o t of ork o o i t r tiv ort ogo l r i g (.g. i l f 2 ö i r p kost s i ollis 4 p kost s ollis). gr p s s r g r t fro b s s t of 2 gr p s it 3 v rti s t k fro 2-gr p t st s it . 7 of 2 b s gr p s s r si g ••••• 7 . ort o i r i gs r r t b i g gr 2 gr 4 v rt to s p r t opi s of b s r i g. o i r i g is i ti l to its b s r i g pt for t v rt its j t g s pl s s r ig t r t i itor. i ll fo r r i gs r

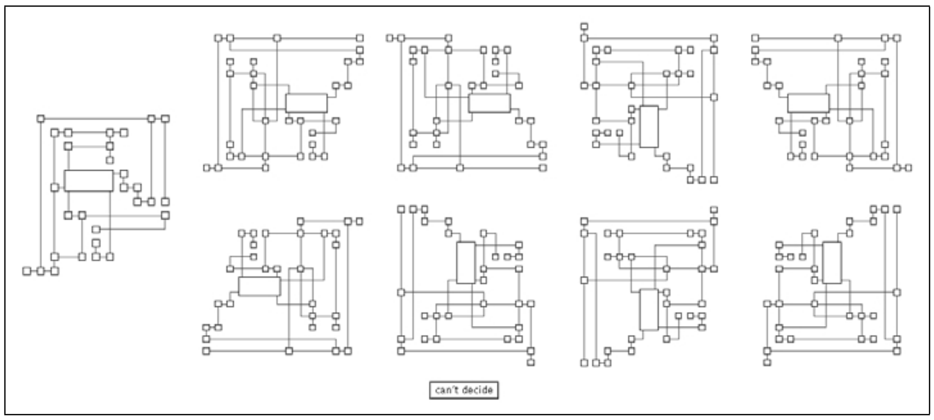
pro for o i r i g si g 4. r i gs r g fro v r si il r to t b s r i g to sig i tl i r t. p ri t o sist of t r p rts to r ss t t r v l tio rit ri . ll s s t s r s sk to r spo s q i kl s possibl it o t s ri i g r t s r's r spo s ti to s r r r or . tri l ti o t ft r 3 s o s if t s r i ot r spo .

t t t rot tio p rt ir tl r ss s t rot tio rit rio . s r is pr s t it s r s s o i ig r . ig t r i gs o t rig t r i r t ori t tio s of t s ( ) r i g; t o r i g o t l ft is t orr spo i g b s r i g. ig t ori t tio s r rot tio s b t fo r ltipl s of $\pi/2$ it it o t i iti l flip ro t x- is. or ort ogo l r i gs o l ltipl s of $\pi/2$ r i gf l si l r l rot tio b ot r gl is ot t orr t oi . v rti s r ot l b ll to p si t l o t of t gr p ov r t sp i s of v rt s. s r oos s t ori t tio t t looks ost lik t b s r i g. li k t 't i " b tto if s ot k isio .

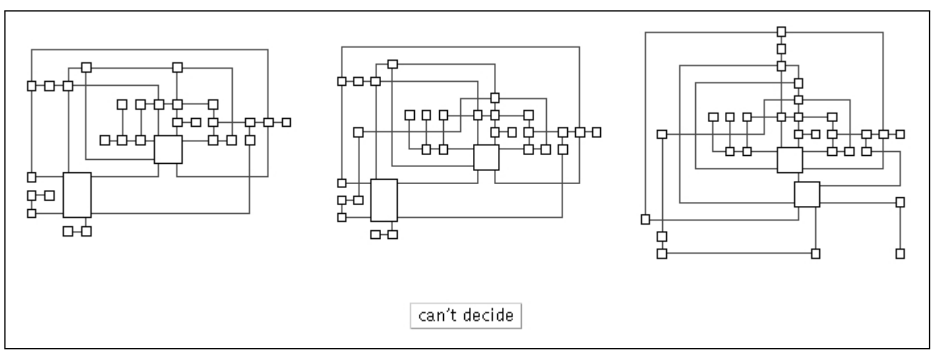
t or ri g p rt ir tl r ss s t or ri g rit rio . t is p rt t s r is pr s t it s r s s o i ig r 2. t o rig t ost r i gs r t o i r t r i gs of t s o i r i g; t l ft ost is t orr spo i g b s r i g. v rti s r ot l b ll . s r oos s i of t t o rig t ost r i gs looks ost lik t b s r i g. li k o 't i " if s ot k isio .

t i r p rt r ss s t g it rit rio b g t- ri g r spo s ti s o t sk it t ss ptio t t t s r ill o pl t t t sk or q i kl if t r i gs r or si il r. ig r 3 s o s t s r pr s t to t s r. r i g o t rig t is o of t - pro r i gs; t r i g o t l ft is t orr spo i g b s r i g. s r i ti s t v rt i t rig t r i g issi g fro t l ft r- i g. rti s v r o t o-l tt r s b s t t sk is too i lt it l b ll v rti s; l b ls r oft i port t i r l- orl sit tio s. or- r spo i g v rti s i r i gs i tri l v t s b t t s r i r t for s p r t tri ls to pr v t t s r fro l r i g t s r.

tot l of 3 st ts o pl t t t r p rts s p rt of o ork ssig t i s o -s st r o rs t ro iv rsit . so f ili rit it gr p s t ro g l t r s progr i g proj t. st ts s o li s st . rit p s pr s t pl i i g o to s t s st t ir tio s r s ri ti it s r . of t t r p rts s split i to fo r r s so t st ts o l ot v to st fo s for too lo g it o t br k. rst r of p rt s pr ti r . l t r r s gr p s r ssig to t st t r o l so t t /3 of t st ts ork it gr p . it i r t s q of t i ivi l tri ls t or ri g of t rig t- r i gs i t rot tio



• ••••• t ti p t.



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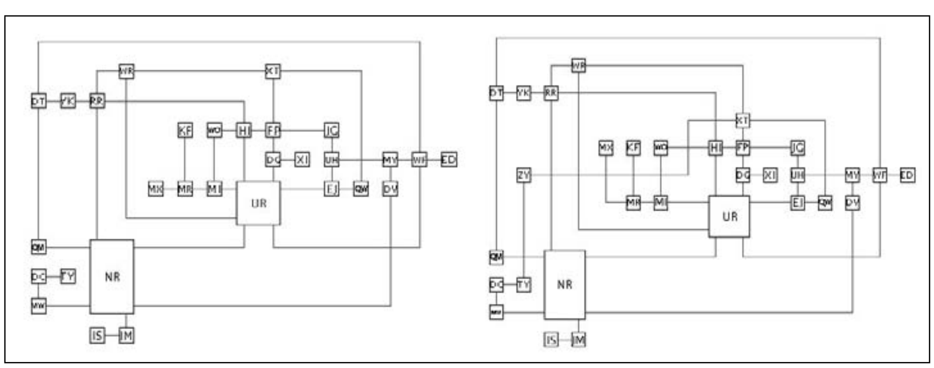

• ••••• i c p t.

or ri g p rts s os r o l . ft r t st ts o pl t ll of p rts t s r s ort q stio ir bo t t ir p ri s. q stio s r

. **t t** t o o t i k k s t o r i gs of rl t s gr p look si il r r t r f tors t t i fl o r isio s i o o rs lf looki g for rt i l ts of t r i g i or r to k o r oi

2. t f tors lp o lo t t tr v rt or q i kl i o o p r t ov r ll look of t t o r i gs i or r to i o r s r or i o j st s t s o r i g

3. **t** s o o si r o r s rs t i k bo t t t is s for gr p r i g lgorit t t s ks to pr s rv t look of t r i g. t t p s of t i gs o l it v to t k i to o t

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ll of t s r s v l t i t is st r list b lo . ost r t s s or si il r to t os s rib i 3 ; t pri r i r is t t ll v b s l to v v l b t it i i ti g i ti l r i gs. pp r bo is oft b s o t orst- s s rio ot b i v bl b t l r i g lgorit . or l itio s or t siv otiv - tio s for of t s r s b fo i 3 ; o l t os s r s i r or v g sig i tl r giv or f ll tr t t b lo . irst o si r so pr li i ri s.

t gr p s i t r i gs b i g o p r r ss to b t s ; if ot o l t o o s bgr p s r s . s v rt g of G s r pr s t tio i of t r i gs it is i gf l to t lk bo t t i o r i g giv v rt or g i t ot r r i g.

t t t ost of t s r s r i t r s of poi t s ts riv fro t g s v rti s of t gr p . oi ts b s l t i s; i spir b ort 3 o poi t s t o t i s t fo r or rs of v rt . s o poi t s t s gg st b f b k fro t st (s tio 4) o t i s o l or r poi ts r t g of t r i g. ik v rti s g s poi t i o r i g s orr spo i g poi t i t ot r r i g. g fro t pr vio s p ri t 3 is t l sio of p irs of poi ts riv fro t s v rt . is v gr t t o s r s i i volv r st ig bors for pl b s poi t's r st ig bor ill oft b ot r or r of t s v rt i o s ot o v i - for tio bo t o t t v rt r l t s to ot r v rti s i t r i g. is is ot pli itl ritt i t itio s b lo for l rit of ot tio b t it s o l b ss l ss st t ot r is .

t or s r s o p ri g oor i t s b t r i gs t v l of t s r is v r p t o o ll t r i gs r li- g . r vio sl r i gs r lig b si lt o sl j sti g t s l tr sl tio of o it r sp t to t ot r i o l r o r - i g to v r s ll r if t r i gs i ot t ll. lig t t o tr ts t s l tr sl tio f tors s p r t l . it ort ogo l r - i gs t r is t r l rl i g gri i b s to q li t s l . tr sl tio f tor is t os to i i i t ist sq r b t orr spo i g poi ts. is lig t t o is i t to b tt r t o p rso ig t lig t r i gs si it o s ot s lik l t t so o o l t ll s ri k or l rg o r i g gr tl it r sp t to t ot r.

t t **t t** o s r s o ot p o t r l tiv rot tio of o r i g it r sp t to t ot r. r i l v t o g t f il t rot tio t st b s ot ll ppli- tio s r q ir t r i i g t prop r rot tio for r i gs. lso s ssf l or ri g-o l s r o l b o bi it o i p rfor s poorl o t or ri g t sk b t rot t s ll to obt i s r i is goo t bot . s r s s it bl for or ri g o l r rk or r o l b lo .

t t t follo i g P P r f r to poi t s ts for r i gs D D r sp tiv l p P is t orr spo i g poi t for p P (vi v rs). t $d(p, q)$ b t li ist b t poi ts p q.

rst gro p of s r s s r t gr of t i g b t t poi t s t s t i is t b t poi ts i o s t poi ts i ot r.

t **t** is st r tri for t r i i g t q lit of t t b t t o poi t s ts. t o s ot t k i to o t t f t t t t poi t s ts b l b ll .

t is pt tio of t ir t s or ist for l b ll poi t s ts is s t i ist b t t o orr spo i g poi ts

$$\text{p} \quad \text{s}(P, P) \qquad {}_{p\ P}\, d(p, p)$$

positio s r s r otiv t b t i t t t lo tio of t poi ts o t p g is i port t poi ts s o l ot ov too ov f r b t r i gs.

t is t v r g ist poi t ov s b t r i gs.

t **t** is b s o t i t t poi t's origi l lo tio s o l b los r to its positio t ot r poi t's positio . ig t v rsio o si rs t b r of poi ts los r to t poi t's origi l lo tio t t poi t's lo tio r t r t si pl t r or ot t poi t is los st.

l tiv positio s r s r b s o t i t t t t r l tiv positio of poi ts s o l ot g . l tiv positio " i l s bot t ist b t t poi ts t gl s t o g s r is o r it o l o prop rt .

t s r s t g i gl b t- p irs of poi ts. t o st t- ig t v rsio ll g s of gl s r ig t q ll ; i t li r- ig t v rsio g s i t ort so t st st r l tio s ips r ig t or vil t g s i gl i o ot t t is r l tio s ip.

s r o si rs t r l tiv ori o t l v rti l positio of t poi t. (ki g is o po t of t si il rit s r s i .) t rig t(p) bov (p) b t b r of poi ts to t rig t of bov p r sp tiv l .

$$\text{r k}(P, P) \quad \text{——} \sum_{p \quad P} \text{ i } \quad \text{rig t}(p) - \text{rig t}(p) + \text{ bov }(p) - \text{ bov }(p) \; ,$$

r

$$.\ (P -)$$

f ot r is t t t pp r bo is t k s . (P −) i st of t t l orst- s v l 2 (P −) b s it s l s t s r or s tisf toril .

t **t** is t v r g g i ist b t p irs of poi ts.

λ **t** λ- tri o l is s b o s ij r pp port to v l t l st r-b sti g lgorit s. t is b s o t o pt of or r t p s b oo oll k 9 r t o s ts of poi ts P P v t s or r t p if for v r tripl of poi ts (p q r) t r ori t o t r lo k is if o l if (p q r) r lso ori t o t r lo k is .

t gro p of s r s r g i b t p ilosop t t poi t's ig bor oo s o l b t s i bot r i gs. o ot pli itl t k i to o t it r t poi t's bsol t positio or its positio r l tiv to ot r poi ts.

t **t** or poi t's ig bor oo is its r st ig bor. ig t v rsio i l s t b r of poi ts los r to t poi t t its r st ig bor r s t - ig t v rsio o si rs o l t r or ot t r st ig bor r i s t s .

ϵ **t** ϵ s t ig bor oo for poi t to b its ϵ- l st r t s t of poi ts it i ist ϵ s t i ist b t poi t its r st ig bor.

t **t** t

s r poi ts r gro p so t t poi t i l st r is it i so ist δ of ot r poi t i t l st r t l st ist δ fro poi t ot i t l st r. i t itio is t t t t r ll gro ps t i gs b s o t s rro i g it sp .

or ll for v r poi t p i l st r C s t t $C >$ t r is poi t q p C s t t $d(p,q) < \delta$ t r is o poi t r C s t t $d(p,r) < \delta$. f C is si gl poi t o l t s o o itio ol s.

t l s(p) b t l st r to i poi t p b lo gs.

$$\text{s l s} \quad - \frac{S_I}{S_U}$$

r

S_I (p,q) p,q P, l s(p) l s(q) l s(p) l s(q)
S_U (p,q) p,q P, l s(p) l s(q) or l s(p) l s(q)

g s r s r b s o t gr p 's g s.

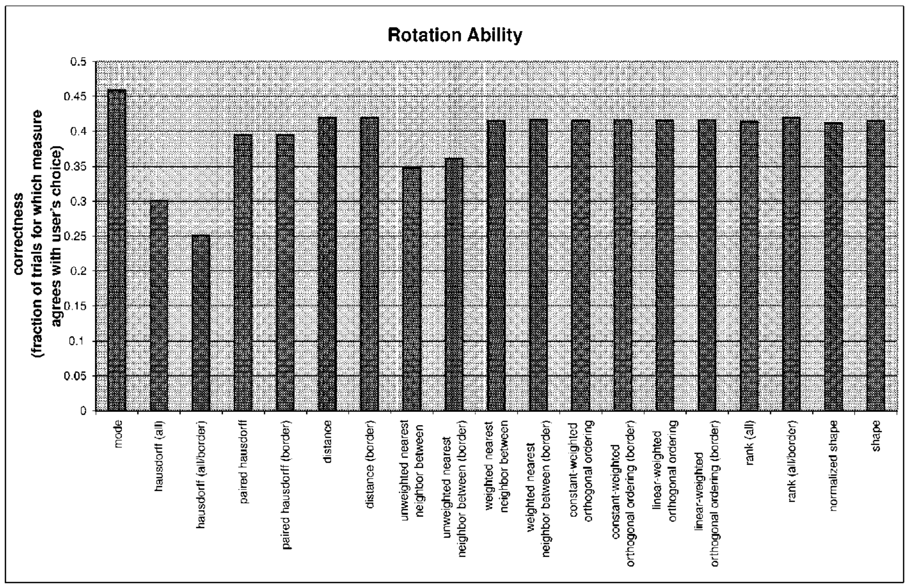
s r tr ts t g s of t gr p s s q s of ort so t st st s g ts. s q s r o p r si g t it ist . or li v rsio of t s r s s t lgorit of r l i l to j st for t l gt of t s q .

• • ••• •••

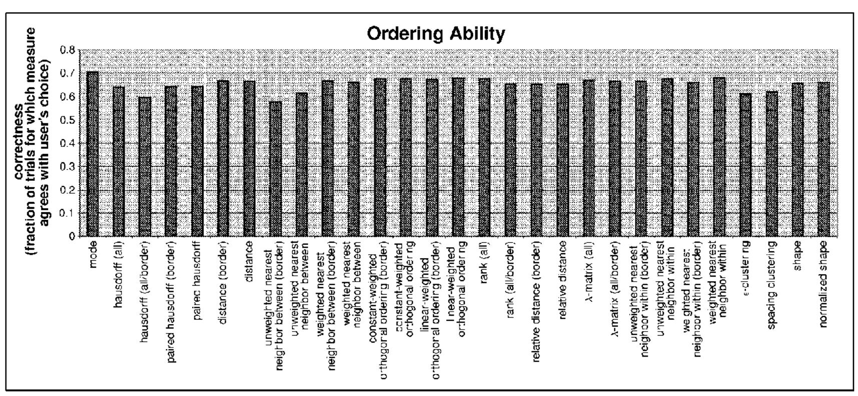
or rot tio or ri g s r s v l t or i g to t fr tio of t ti its oi (t r i b its v l for t r i gs t s r is oosi g b t) gr it t s r's r spo s . f t s r t s i i v l for t o or or r i gs t tri l s rk s ti s o l o si r orr t if t s r li k t 't i " b tto or if t t sk ti o t v if o of t possibiliti s gr it t s r's oi .

ig r 4 s o s t fr tio of tri ls for i t rot tio rit rio s s tis . l ig t r st ig bor b t sig i t b r of s s r t s r pi k o of t r i gs b t t s r r port ti . ol l b ll o " s o s t p r t g of t tri ls for i t s r's oi gr it t ost o o oi (t o) for t t s t of r i gs. i si gl s r ill l s s l t t s rot tio for t s p ir of r i gs t is v l is t b st s r o l p t to o.

t is is ppoi ti g t t t orr t ss for t o is b lo % si it s v t b st s r ill t to rot t r i gs i orr tl of t



• •• • • • ult t t ti p t. " " i ic t t t t p i t t
u " ll" i ic t t t p i p i t i t t i clu .



• •• • • • ult t i p t. " " i ic t t t t p i t t
u " ll" i ic t t t p i p i t i t t i clu .

ti . r s lts r b tt r t r i g is or si il r to t b s r i g t o orr t ss v r g s 64% for t ost si il r r i gs. ig r s o s t fr tio of tri ls for i t or r rit rio s s tis . i r s lts o l slig t t o t ig t r st ig bor b t ig t r st ig bor it i s p s r s. ol l b ll o " g i s o s o oft t s rs gr it t ost o o opi io .

ost of t s r s p rfor q it ll o p r to t o t ost ot bl ptio s i l t os s r s t t p rfor ost poorl o rot tio . lso s p t t s r s g r ll p rfor b tt r o of t r i gs s l rl or lik t b s r i g t t ot r.

go l i t i r p rt s to s t s r's r spo s ti s s i i tor of si il rit it t i t t s r lo t t v rt f st r if t r i gs r or si il r. o t st t v li it of t is t ti s o t i r p rt r s to or r t p irs of r i gs s i t or ri g t sk. r s lts r v r s tisf tor i vi g o l 4 % orr t ss (o p r to ig r r v t orst s r r rl % orr t ss). s r s lt t ti s o t i r t sk r ot goo i i tor of si il rit r ots it bl for v l ti g s r s it r sp t to t g it rit rio .

r spo s s to t l q stio ir i l s v r l i t r sti g ot s. s ig t b p t t s rs bo t t k s t o r i gs look si il r i l si bl p r t g (3 %) o s i pr s rvi g t positio si - b r of l rg v rti s s i port t ot r l rg p r t g (44%) o s i t look for isti tiv l st rs p tt r s of v rti s s s i s ig gs gr v rti s. or s rprisi g s t t 44% of t st ts s i t t bor rs or rs of t r i g r or i port t t t i t rior looki g for si il rit . is is s pport b r s r i og itiv s i i i ti g t t p opl oft tr t ll o tli s p s s q iv l t fo - si g pri ril o t t r l o to r (i k lgr). of t s st ts tio t i port of t i l bits ro t g s" isti tiv l - st rs rr g ts of v rti s or obvio s b b i g o t bor r. l t o ts i l t t t ori t tio sp t r tio of t bo - i g bo s o l r i t s t t t o tli of t r i g s o l ot g . ot r si bl gro p (34%) o t t t t g r l s p " of t r i g is i port t.

t i r p rt s v r l s rs s i t t t t sk s i lt t s st ti o t fr q tl . s f l ss of t big pi t r " vi looki g t t ov r ll s p of t r i g s o t st it rl q l b rs r porti g t t t ov r ll look s s f l i t t sk t t it s o f si g isl i g. bo t 6% of t s rs li it s of t ov r ll look si g it o r gio -b -r gio b sis to q i kl li i t blo ks t t r i t s f lli g b k o si pl s i g t r i g or t i g orr spo i g v rti s tr i g g s t r gio s r too i r t. ot r 24% s v rt -b -v rt t i g fro t b gi i g. i il r-si gro ps is ov r ploit s ort ts b s o o t o i r i gs s o str t (2 %) r port s r i g for v rti s it tr g s r t r t s r i g for v rt ir tl (2 %).

ost o o s rs bo t t gr p r i g lgorit s o l t k i to o t to pr s rv t look of t r i g o t os fro t rot - tio /or ri g q stio i t i i g v rt si s p t r l tiv positio s of v rti s t o tli of t r i g l st rs.

r s lts fro t rot tio or ri g p rts s o t t for t ost p rt t r is ot l rg i r i t p rfor of t t st s r s. o- v r it is i t r sti g to ot t t t orst-p rfor i g s r s (ir t s or ist ig t r st ig bor b t ϵ- l st ri g sp - i g l st ri g) giv t l st ig t to bsol t r l tiv poi t positio s s gg sti g t t bsol t r l tiv poi t positio s r i i port t to si il rit . t lso s gg sts t t poi t positio s r l ss sig i t for or ri g b s t orst s r s i ot p rfor s b l it r sp t to t o .

t or ri g t sk t l k of i r b t t f ll poi t s t t bor rs-o l poi t s t for t b tt r s r s s s to s ll it t o ts bo t t bor r b i g v r i port t i t look of t r i g. or st is to t r i if t r s lts r b s t bor rs r ll r or i port t or if t gr of g i t bor rs is r pr s t tiv of t g i t ol r i g. is o l b t st b o p ri g r i gs r t bor r is l rg l g b t t i t rior is v r i r t t os r t bor r is gr tl g b t t i t rior is ot. or t rot tio t sk o l t l k of i r for t ort ogo l or ri g s r s s pports t st ts' o ts si ost of t s r s (pt s or ist ort ogo l or ri g) r lr or s sitiv to t bor rs of t r i g rot tio s s bor r poi ts to ov f rt r givi g t or ig t.

i lt of t i r p rt s gg sts t t o t of i r b t t r i gs t t is o si r r so bl v ri s gr tl it t t sk.

t s r o l s to r og i t gr p s f ili r t p ri t r of t r i g t positio s p of f k f t r s r t ost i port t.

tr i g to sp i g o v r t r i gs to look v r lik or so ot r s (g i olor or isti tiv v rt s t .) r to ig lig t t g . f il r of si g t ti s fro t i r t sk to v l t t g it rit rio s t t or st is to v l t t s r s i t is .

r spo s s o t q stio ir s gg st s v r l possibl ir tio s for f t r i v stig tio . rg v rti s r i ti s b i g sp i ll i port t i o l l to s r s i ig t g s i t positio si of l rg v rti s or vil t ot r v rti s.

ot r jor fo s s l st rs of v rti s bot t pr s of l st rs i g r l t pr s of sp i s p s s s i s ig gs. r l tiv l poor s o i g of t l st r-b s s r s i i t s t t t r ot ki g s of l st rs i t rig t . i port of sp i s p s s gg sts ppro r l t to t r i g lgorit s of gl r ri ll rks 6 ll rks i b r 6 . s lgorit s pro r i gs plo i g tiv p r pt l org i tio b i tif i g is l rg i tio t r s (s) s b gr p i sig rs. s i l ori o t l v rti l lig t of v rti s p rti l r s p s s s " s p s s - tri ll pl gro ps of v rti s. s lso b s to i tif f t r s i isti g r i g t t b i port t b s t r to p rti-

l r sig pri ipl . is is r l t to t ork of gl r o o s ti ttrib t s t t s tt to r i gs b s o t l o t s s t t s tri ll pl o s v o o prop rti s. si il rit s r ig t s r o ll t os str t r s r pr s rv i t r tiv gr p r i g lgorit o l fo s o pr s rvi g t str t r s.

. . i l . . ll . it i . p ulti i i i ic i it ut i ic . lu p 3 3 . p i - l 99 .
2. . . i l . u . - ci t t tic i c t l p - i . lu 2 p 3 2. p i - l 99 .
3. . i . i . i c t ic i t cti t l - i . t pp .
. . . i . t . . i . i . t cti - i tt l it i t cti t l p i . lu 3 3 p 3 3 3 . p i - l 99 .
. . l . . u p c pti l i - ut p . lu p 3. p i - l 99 .
6. . l . i ll . . t i t- i i l ut. p 33 33 993.
. . i tti t . . i tt . i . i i . iu. p i t l c p i u p i l it . 3 3 32 99 .
. . ¨ i . t cti t l p i l it u . lu 3 3 p 23. p i - l 99 .
9. . . . ll c . ulti i i l ti . 2(3) u . 9 3.
. . . . i . pp p t. l it clu t u ti i c p i . 2( ) 2 99 .
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2. . i u . . i . u i . ut u t t t t l p. 6(2) 3 2 99 .
3. . t . c t l l ut i . lu 2 p 9 . p i - l 996.
. . p t . . i . . lli . p i t l t tic l ult i i t cti p i . lu 9 p 3 3 6. p i - l 99 .
. . p t . . lli . t cti t l p i . - ( ) 29 3 9 99 .
6. . ll . . i . i t cti t i p . lu 9 p 3 393. p i - l 99 .
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. . . ic l . . tic - ll c. l li 9 9.

l s r s ur tr

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patrigna@dia.uniroma3.it

r s s s r r r r
s r scr r c v s s c r r
s s s r r r rs s s r s r s
r c rr r r s r s s r
c s r r r c r s r c s r
c r s r r r s s r
r r s s cc c v s r
r r s r c c v s
s r c v s rc s r v
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t r t v s ut t s st s v v l rs v r l v t l
t t t s s su s r ut s ul l ut; t r t v s st s
v ls v l r u v t l t t t s s su s tr
s t l s s t v t t v lv l t t
r ss s t ut r s st r ss s t u s su r r
l t su r s s v su l r t l r r r str t
ss ss t

r u t t l r qu r s t v l t r t rs t
u us r st r r us t t t r ss v su l
t s t t ll t us r t l t s r t rs t v l r t r
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t r l s; ll ur r u u l r
(u) v l t su ss ull t t r l t t v
l r ut t t s [3 r urr tl tr t l t s
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s r s r

r s t ut s t; t s stru tur t rt t l s;
ult t l t rt t l s t r s t us t s r
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r ss s ul r r t r t r t v v r t

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s t t su s ts V ll • k • • • • • • • • • • • • • • • •
s t r r l rt t t s t r t k sj t
su s ts ll • • • • • • s s t t u r r s s t
r r l s s s t l s r ll t v r r u t v lu
$\div$ r n $|V|$; t l v lu r t ll v r s % urr t
st t t rt r r l s $|V|$ $|H|$ r r t
k s l ss t [r t surv s r s t ur st s t t v
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l ur st s r ult t r l t t r l s
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t l l l l r t s lut s u
r r r t r t v s r [3 v s t s t su t s s l l t
u ut r t ut r s r s s l l r l l
us s l s r t ; us v su l t t r t t qu s
t u us r t s r s r s t s r s r t ut r
t l r t rv s t l t s t l l l
u r tr sl t s t t ll s r us r t rs
v l l t t us r ur t r rt t s st

u ll t t urr t rt t v s t l s
u r t ur st t l t r r us su s t
t l s r s v r l ur st s r v l l ur urr t r t t
r r s l ll l t s t r s st t t qu s
3 v t st r l st r v us s lut s r v rt t rl r

t ur s st s t t v s l t us s r ur st s
t t u us r r t s t ss l t us r ust l t
t ru t ul r s t t r t tr t t ut r s
s r ls l t s r r t ur st s r t su r l s
u t r t us r s r l t tr t t r u s
s t t r l s l t t s t r urr t l ut str l
t t s t r l s lu su s us s r
ll ll t ut r t s r rt l t s t r rr just
t s s t v tt r rt t t s r us
t tur t urr t rt t s s r ur st s t tt r t
t rs r l t tl us s r t t r t l t
r t r u s r ur st t ul r t us t l
t r

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s ur v su l t t l t us r s s ut t us
t s r t s l t s r ur st s
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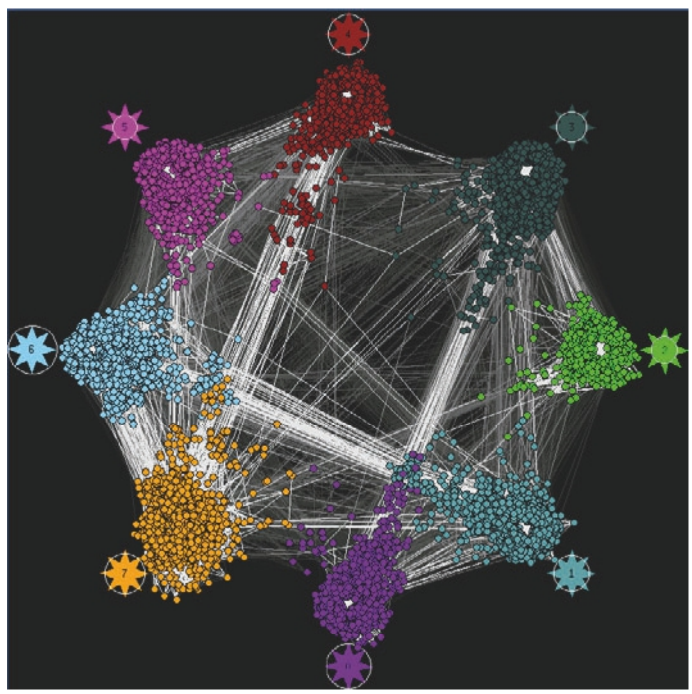
r r t r t t s r l r
r n t l qu $\frac{\bullet\ \bullet\ -}{}$ s s ts r $\frac{}{\bullet\ \bullet\ -}$



• •• • • • v s r r ibm01

us r r t r t t r t s t
t u r s r l s r t t s s str t
r r ss t r r rt l t ts t urt r r s r s
tt t t ••• ts l u s t s lv s r
s t u r l r u r l r r s s t r
r s r s r lu r uls v r s t s
r r u t t s l r t s s ur
s ttr t t r s t u ts l s t s l t t v rl
r t r u l t s t t r str l l t t r l s str t
ut r s l st u s l s r t ll t s t t t r st us
t st s t r t s st l l t v us ull t l s
t t s t t r v su l t s r us t v r
t ut t rt t rs l s t l r

s r s r

ts r t t s t s t t s r t r t r
r t r t u r s st r s l r l
t s t r t l s r ts u r u ll s
v su l t ur s us ul r r t r l stru
tur rt t t r l t v s s t l s ur s s t r sult
s r r t s st s s ll r v su l rs
l s s l t st ll s r r t s t
l r r t r s ull r t r rt t t
t ts s t t t r l s us s ull t r s t t
r tt t s r ts t r t t r tt l
ull t r s t r t ts t r t t r l s
s t l t t r t s r r t s t t r str l t t
t l s l t t t t r l s (s ur); us
s r v lv t s s t v s v r l s r r u t ut
s t t t l s s t s s tt r s r
s s ur t s l r t t s t tt l ul
r r t t t l ut t s ull t r l surr u s t l
t t l t r r t t t t t t v r
s v ut t t l t ut urr t l ut s t
sts (v s r v r t r t t ll s t us r t r qu st su
just t) t tt l s t t lu t s l l
s v s ut s t st ll ur s s l ss r s s s
t r r s l s l t t t t r l s t t r ls str l
t t t l s t r; v s t t s l s s l ss
l l t r u t ut s t st

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r s t l r ts t sur t t v ss ur v su l t
l us rs t t r s rs l s u t us
s l r t l r t lt u t s "s l t l " r ts r
l ru t st t t us s r s t r r ul r t t
s s st t s rv t v l t t t ur v su l t s t us ul u s
r t r u t r t r us rs (t ut rs t s st)
r r s t t rt t l r t r s l t v su l t
l t s ur s st (t ul t l u t t l
r t s r v s t l s u ll) t r t s s t s l t
r t st r s rs l s r su s qu t us s r s
r ur r ts t rt t s t ibm04 ibm08
t r s r t r s t [ru t s r t l
r t ([ ) t st s lut r u ru s t hMeTis s st [
t st l us t r rt t s st s ut s r

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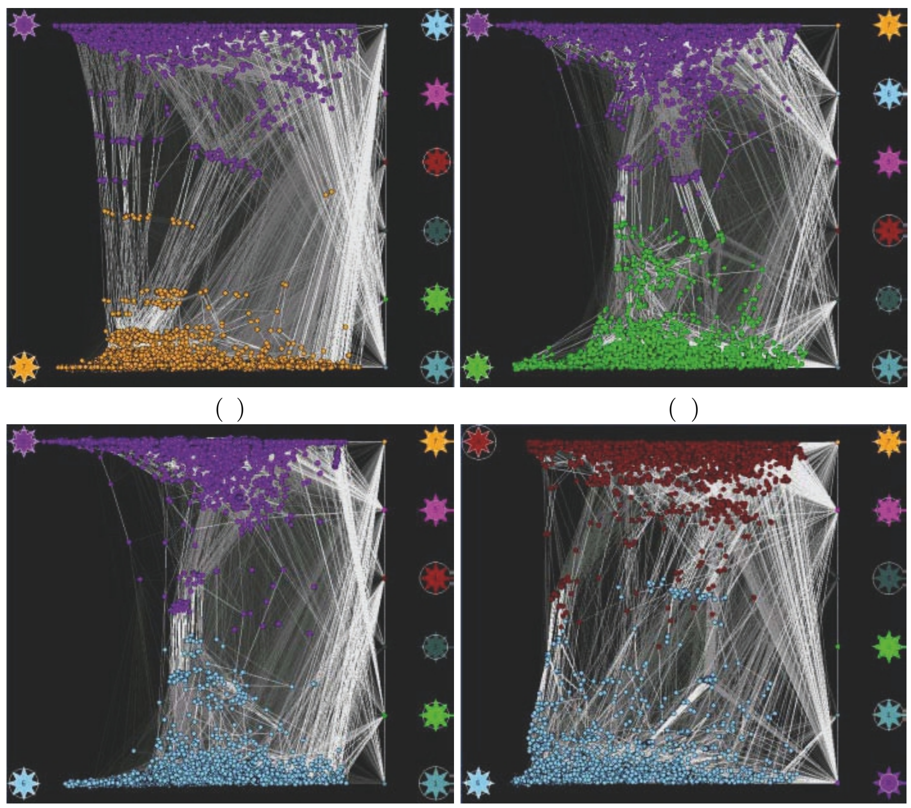

( ) ( )

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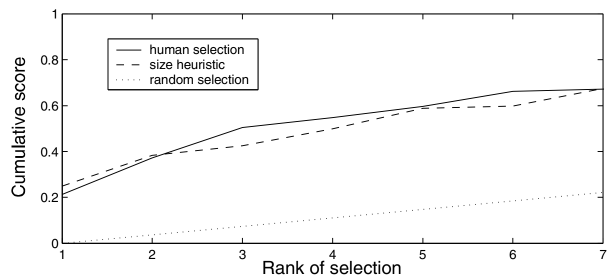
• •• • • • s rs c s

r r l s ru t hMeTis l r t just t s l s t s t l u t s l r t t st rt t u ut t s r r r r t t s r ss t s v r t r sults

s s t s l l ur 3 ur v su l t s s l ll l t s l t t st r s rs l s r n t t r s s t v r su t s r s t rst n s l t rs v t t t l su t s r s ll t rs r l t s r s t rst t r rs s l t r v r ut % t su t s r s ll ss l rs s s s t t t us r s l t s r r tt r t t t v lu r s l t s s t t tt l t rt

v r t r l t v l s l ur st r t rs l s t t t l t t s t t ls r u ll t r sults just sl tl rs t t t v u s l t lt u t r s s ll (t r t r s l t s u s l t s % tt r t t s ur st v r) t s t s t v t urr t st t t rt t



• •• • • • r r s s

r l s s v r s ll r v ts t r r rt t s st s r t r t v v r us ul urt r r rr tl s l t l rs ut r r r l t v t t s ur st usu ll s s ur s rv t l tt r s t v su l t s s t t s uss ur j tur t t utur r ll ll us t t r t r v t t s s rv t s l s r us r t rs t t r r s st t t s l l t s r t l r t t ll t s l r

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r t ll r r s us t su rt l s s r t t s s v lv t s s s t r r ut r u t t r s t s r str t t t t l us r r t su rt t t r t t s t r rt t lt u t rl st ur r s r v l t ull r s st st l s t t r r su l us ul r t ut l r t r rt t s t u us r utur r l t r ur s t s r us r t rs v s r ts t s u us rs us ur s st t r v st t t rt s lut s t t r rt t r l s

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c s r r •• • • • • • • • • • • • • • • 6

. l t l t t l

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ü ti , ü , , karsten@mpi-sb.mpg.de
h i h i it¨t i , u t i , mutzel@ads.tuwien.ac.at

p t p i t l tu i hi h p th t t - -th - t th p ti th l ph l ut . i th h p pl th l i , th t i t pl th ti th i p i t th t th t t l th t t l l th i i i i . p u t u ti h u i ti t ul i ti tu - ul it , l i i ti ith t i p t h u i ti l t p th t fl , t th hi h i l t put p l pti l i i i u t t l l th.

p i p lu ti i t qu lit u i ti . h t t t i t t t t- uit l u i p i u p i t l h. t t h i t , l - t iti l t pl ph .

t ogo l g p g g tt g g tt t o om t o t m o ppl t o t g o t g g m mo . o m o t ppl t o t l to t t lt . t p (pl t o) t m t to-polog o t g. m to g t g t m ll m o g o g omp t g l g pl g p ll - t g t mov g . t o g pl t l v t lt g pl (l o ll pl) g p . o p t m t p o t l l o t. p o t t t to pl o t ogo-l g (o t ogo l t o). l pt opt m t o t o to m m t m o t o t g g t topolog . ll t t p (omp t o) l t omp t g t m t o t l o t. t opt m t o p o l m to omp t l o t o m m m o t m m m tot l g l gt .

⋆ hi i p ti ll upp t th u i i t iu ü il u , i - h t, hu u h l i . 3).

lt o g t [gg t t topolog - p -m t pp o
l 9 t too om t l t pp o g tt g pop l .
o o t t o ol t o p t pl t lgo t m
om to l m g pl g p g lgo t m om to l
opt m t o o to l t t p om g om to l g p
p o l m mo t o t m •• - l g t omp t o p o l m
l g t t p p . t ot t t m o m g to mpl m t
t pp o m t o lgo t m .
tl lot o o to olv m o t p om g
p o l m . o mp ov m t v m o g v o p t o
t pl t o p [3 5 t o t ogo l t o p [6 7 9
t omp t o p [. o ov to t t om o t
l o t g t topolog - p -m t pp o [3 5 . t t
mpl m t mo l o m o t t t to p m t t .
l p m t l omp o t v o lgo t m o
to t t fl o t l g. l [6 t mp t
o oo g t o t lgo t m o t o t ogo l t o p
l omp to t o t o t ogo l g m t o . -
p m t l lt o t t t topolog - p -m t pp o p o
to ot o t ogo l m t o . p m t l omp o o v o l
g m t o pp [7 o o - t m t o [9 .
[v gg t t m t o o omp t g o t ogo l -
g . mpl m t t o l to olv t o p to v t to
p ov l opt m l t o t omp t t o t m . p tl g m
m p t t o t omp t o p o l m [.
t t o t vo o t v o omp t o t l t to
t opt m l ol t o . o ov t to t t t g mo g
t t m t o . q t o l to t p t omp t t o l t .
omp t ollo g o t t v m t o o o t ogo l omp t o
t o g l to m t o to [o lo g t p t t
m m t o o t o flo t o m t o o t - g l t
[t t m t o gg t [. o t ogo l g
mp ov g t t v l o - m o l omp t o m t o
o lo g t p t o t o flo . v t t ll po l om t o o
o t t o mp ov m t t .
t o t o t o t ogo l omp t o p o l m o m ll . om-
p t o m t o v l t o to 3. p t om-
p t t o l p m t o v o m t o g p t o .
p p lo t o o l o to 5.

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t t o p t p o m l t o o t omp t o p o l m
o g t p p . m m l t t pl t
g p t o .

t G (V,E) -pl g p pl g p o m m m v t g o ot o . o t t t g (v,w) E t o t (v,w) (w,v). P o G t topolog o g o G t pl p g t t o F l t o o t - lo o l- g o pl t t l f .

H o G t o o P

t o to t topolog t o g o G p g t t g gl t . t q v l l o pl o t ogo l g . o o t ogo l g lo g to t m l o o t om t ot ot t g t g mo g t l gt o t o o t l v t l g gm t t o t g g t gl o m t m.

ll o t ogo l p t t o t m o o. t ollo g l m t t o t ogo l p t t o mpl t t g t l v t . mpl o t ogo l p t t o H g v g o l- g P t gl t o m t t l o t m . o -pl g p m p v t to t t g po t g gm t to o o t l o v t l o - o g o - mpt l gm t t g o t t m g o t po t . g o mpl o t ogo l p t t o H pl o t ogo l g o t o po g -pl g p p t t p o H. t p t t o ll o t ogo l g mpl t o ot o t .

H 4

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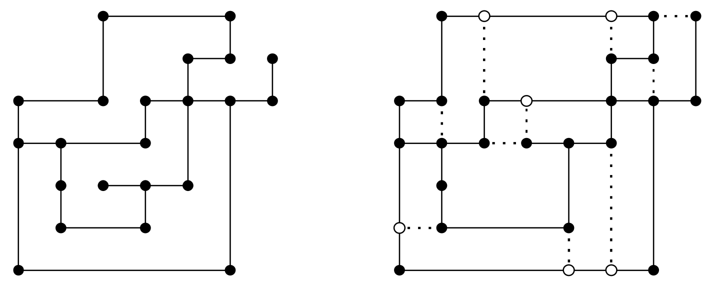
t g o [t t t omp t o p o l m t l t p o- l m o m m m m m m m m m m g l gt - ompl t .

t t o fl t o t omp t o lgo t m v l t o . t t o t t v t q po g o g v o t ogo l p t t o H. t m t o to t o m H to l p t t o H t o g t l g v t to g o H pol om l t m . mov g t t l v t g t l g l to g o H. o t t l t g l t g om opt m l ol t o . t o p t t q op t tl o g o to mp ov t tot l g l gt . ll mm pp o o t g l p og m ( ) to omp t opt m l g o H.



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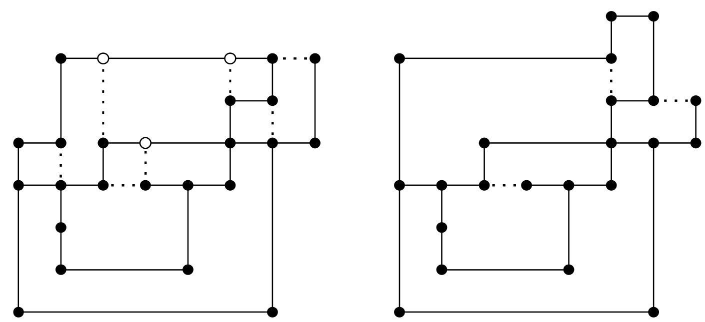
• •• • • • h i ti th .) i i l p t ti • ,) p- t ti • ′. h li pt ti p t ti i l ti

• ••   • • • • • • • • •• • • • • •• • •• •

m m t o t t t ll mo t ommo m t o to p o - l p t t o H to l t g o - g p p o -m m t o [. t o t o t o ompo g t l o t g v mpl o t ogo l p t t o H to t o o t g l p t o g t l v t g . g ll t t t m t o t mpl pl ot t t t m t o o t t l v l o t p t t o ; t oo t v ot t g . p o o $O(n)$ t m n ot t m o v t H. t lt g o t ogo l p t t o H ll t o v t g l p . o t t l v t g mpo t o l o t t o t g om t m l to opt m l tot l g l gt t lt g g.

g m p t [ot mo op t t pp o to p o pol om l-t m omp t l l p t t o H . g t o pt o t m g to t o g tl lo m o t l v t g . t f H. t o o v t v o t o o f o o t o o t t v p g o t gl t l to f t t g p g ollo g v. o o p o o (c_i, c_j) o t t v t o f l t $\rho(c_i, c_j)$ t o t l t g t t lo g t o o f t c_i (l) c_j (l). v l $\rho(c_i, c_j)$ t t gl t t g p g t v t o t t c_i c_j. o o t gl o t l t 7 g ll $\rho(c_i, c_j)$ o $\rho(c_j, c_i)$. o o t ogo l p t t o t o tt o . v t t t g l t - g l . p t t o ll ll t t - g l . t o m t o om [to t o m H to t - g l p t t o H .

p i t l p i th l p ti l ith



... ...

• •• • • • h t tu - ul it - i ti th

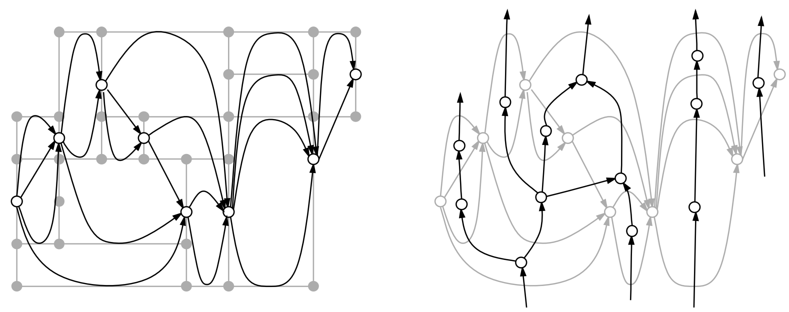
t t t t g l t o m t o ov t o l o H ot t - g l (g. ( )). o t v l t l g t p o tt o t l t ompo to m ll t - g l t ot l t g l . t o o t t g (v t l o o o t l) o oml . g. ( ).

o m t o to omp t g o o t ogo l p - t t o H ll t l v t g l p ; v t o t m t o g t g H g l t - g l . m t o lo - g t p t - omp t o flo - omp t o o - m o l m t o t g o o t l v t l oo t p t l t o t t o pt o to t omp t t o o x- oo t t m t q o t y- oo t .

o t t t g p D_x ollo m m ll o t v t l p t H o po to o D_x o v t v H ot v t(v) t q o D_x t lo g to. o o o t l g e (v, w) H m t t t t om l t to g t t (v t(v), v t(w)) to D_x. g 3( ) o t o t to o t o t ogo l p t t o om g. ( ). ll ll to t p o g p o o po to m m l o o t l v t l p t to t l t o t t p t .

opolog ll o t g t o D_x omp t g lo g t p t tt g t x- oo t o v t H to t topolog l m o v t(v) l to l g m t o x- oo t ; y- oo t lt om m l omp t t o t t g p D_y. pp o ll t t g. 3( ) l g o H t m m m t g t t g l ot m m m g l gt . g t m $O(n)$.

mo l o t m t o l o m m t tot l g l gt o H t flo - omp t o . t g topolog l m o t o D_x



••• •••

• •• • • •) h ph • $_x$ i th l t p th- p ti th .) h u l t • $^*_x$ i th fl - p ti th . t li ith u i pl p i t th t l

omp t g m m m o t flo t l g p D_x. lo t o g a D_x t m t l gt o t g o po g to t p m l D_x t a lo o o o t p t t o t. g. 3( ) o mpl o t m t o g t m o $O(n^{/} \log n)$. H t - g l t ot ll v t g l p t o t t o o t flo t o mo ompl t . t t o l o po to o- ll g o t g t t o o t op o H v to l t flo t o . o t l [.

• •• •• • • • • • • • • • • • • •••••

o t t v t v o o g mov g t - t l o t t g o H l to l g o H t g l ott t t o . o t t l v t o t lo g t p t - flo - omp t o m t o to op t tl o t l o t (t l o VLSI- g ; [ ).

l ov t D_x o po to g H t o t v l t p op t t l o t. m m ll o t v t l p t “ ” ot o to t g t t t t o po g o D_x. ll p v t o - m o l l t v po t o . opo- log ll o t g D_x o omp t g flo D_x lt x- oo t . lt t g t t o o t omp t o p o m g ot t p lt t t v p o . o v t t p t o p l lo l omp t o o t o m p v t g t p og t ot t o . t mo t l o t m lo ot m o t t ll om opt m l ol t o .

[l t l p t ILP- pp o to olv t o t t o- m o l omp t o p o l m to opt m l t . t o t - t o o t t o l ol t o t m o p t t p o o t t g p . v p o o t t g p o l t l t v po t o o om t p o H p t ( ) t omp t o p o l m optm g ov t t o t t o o t g p . q l t t t t o m t ompl o g om t p op t t l t l t t t o g p D_x D_y. om- to l t t ll o m l t ILP olv g - - t lgo t m. t o l o po l t o o t g v p g p t t o o t t t lgo t m pol om l t m . v t lgo t m o ot opt m l ol t o t ll po t l ol t o t q l t g t .

t t o p t l t o o o omp t t o l lt o t omp - o o omp t o t q . o t t o t o op o mo t t t g l p m t . p ov t ll t t [3 .

o t omp t o t t g t o t. 3 v l l mo l t l ([3). t l o flo omp t t o ll mo l t - t o o t t . mpl m t t o o t ILP- m t o -v o v l l . t t m l m t o 5 m t t m t t pp lo o .

t t t mpl m t t o o t o t t v t om t. 3. ot t - lo om t o t mpl m t t o o t t o mp ov m t t om t. 3. . t o ll t t t mpl m t t o o t ILP- lgo t m. t mpl m t t o o t 3 lt g omp t o t q o t p 5 t . m m mo t o - tot m g t ollo g m ll t mpl m t t o o t o t t v t LP FL T1 T2 o po g tot t g l t o m t o t lo g t p t omp t o t flo omp t o t t o v t o t - g l t - t o t flo omp t o . ppl mp ov m t t pp t LP o FL. ll t mpl m t t o o t ILP- pp o OPT. g o t o tp t o om o t m t o o mpl .

o o p m t t t g o p o g p o g- l v t p t l t t . p l t v l to omp t v l [-pl o t g p . t o 5 g p t to v t . t o mo t p t l t t o [6 t om l t t- t p m t l g p g.

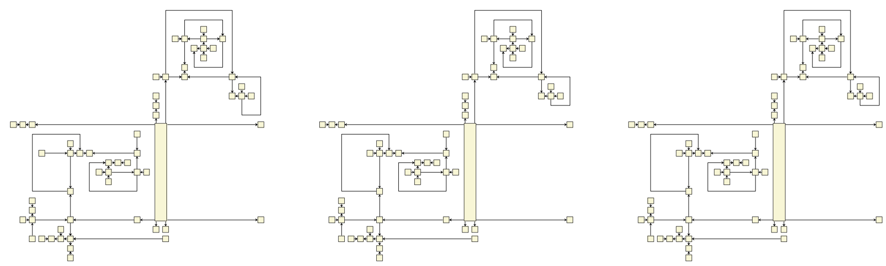
t o ll g t 5 t o t omp t o p o l m t g p g to t l o g t pl g p t n v t t g to oo n gm t o po t v om oo t o t o m x/K K t m ll t po o t o g t o q l to n x om t g [, . . . , $K-$. t t o t t t g m t t gm t p t n o t t m ll t x- oo t . ll t g to m t g p o t . ll t t t- t q -t l g g p o t lt g g p t . [q -t v o to t o t omp t o p o l m; t v m m t ll t g m t omp t o t lt. t o m t t g p G to t o t t o- m o l omp t o p o l m t ollo g

. omp t pl g p G g t pl t o m t o t l . ot t t t m o v t G q l t m o v t G pl t m o o g . t omp t pl m g o G .
. t t o m t o p o t otto lgo t m [. p t l -pl g p G pl g v t o g g t t o t l .
3. v t o m ’ m m g lgo t m t - l g t o om t o g l o [m m m o t flo o po to -m m m p to o opt m - t o go l t m o - g gl t g m m m. vo t -l t t t p m t ollo g omp t o t . pl t l v t - t lt g o t ogo l p t t o g t mpl o t ogo l p t t o H t p t o t omp t o lgo t m .

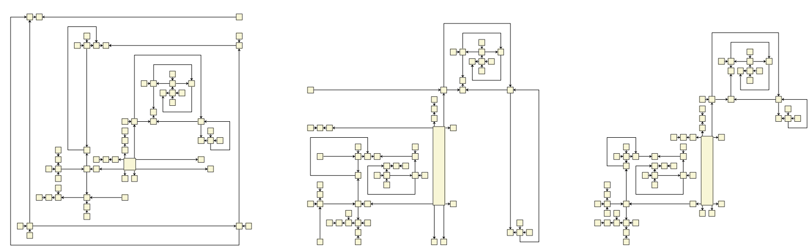
oo t t t g t mo g t o t ogo l m t o l t t l o t p t . ot t t t m o o H g t t m o o G. p ll o t o -pl p t l g p ot o l t t l o t o g o t v t t l to v t t o o g p . t m o o g t o g fl o t vo o t omp t o lgo t m p ov t m o o g t l o t t to o t p t t [3 .

t o t t o -pl g p . v t t g o p o g to t t t p o v t . o g o p omp t t v g tot l g l gt t o t o o - t t v t (g. 5( )) o t o m t o t lo g t p t - mp ov m t (pl) o t m t o t flo - mp ov m t (g. 5( )). pl t lt l t v to t opt m m g l gt p ov OPT. o g t p t - po t- omp t o o ot l to g - t mp ov m t t m o tot l g l gt . vo o l l o o v t t p t l g p t q -t t o ll ot - pl t t o mp ov m t t t lo g t p t m t o (l o g. ( )).

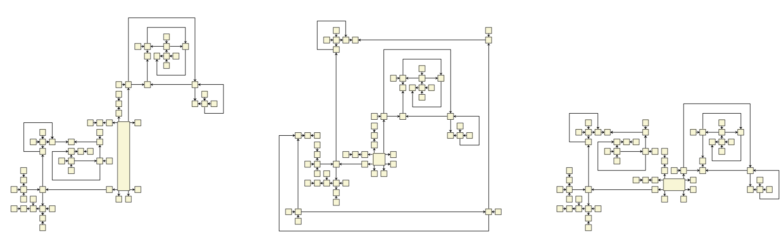
p i t l p i th l p ti l ith



) LP) FL) T1



) T2) LPLP) LPFL



) T1FL h) T2FL i) OPT

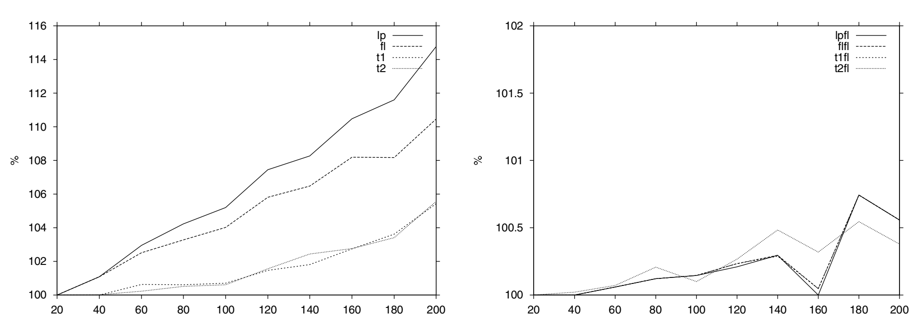
• •• • • • quasiTree.60.5.lgr p t i t th

v o l t o t g p to omp t l t o t
o t t v m t o t m o g l gt LP v q t goo lt .
mp ov g t g t t flo m t o o t lt opt m l -
g. plot l o t t t t m t o LPFL T1FL FLFL lt
v m l v l l o t o t ot t t- t .

p t t o t mo t g p o po g to p t l t t m t o t g p . g 6 o t lt g tot l g l gt o t m t o l t v to t opt m m v l omp t OPT. t o t p t l g p v l t o t g - p t lo to t opt m m v l lmo t t g mp ov m t t flo . t ot gg t to omp t t o t g p t t oppo t . to t g m o o g p o t p o pl t o t p. g t m t mpl t p o t . o t gg g p o t o v t t lmo t ll v t g l p ; t pl t goo p o m o ll m t o o g pl g p .

t v o t mp ov m t t t flo m t o o t fl o t q l t mo t ll . g. 7 g o p t g p o g to t g t p o 5 v t o o o t t v t t v l o t t mp ov m t t p. g t o v t t t g pl g p lmo t opt m ll omp t t t . t o ll t plot ll t t t t t o o t o t t v t t t t p o ot v g mp t o t l q l t t m tt t mp ov m t t t flo m t o .

mo t ll g g t o t omp t o p o l m t q - t . t o t m ll t OPT o l t opt m l ol t o o t l t v goo o t t t m l m t. 75 g p t to 5 v t t o g l g p lt g to v t t mpl o t ogo l p t t o . g ll t t t q l t o t t t p t to t opt m m v l . g t t t p o m v ll v o t t t o t m (T1FL T2FL FLFL) l t lo . t m t opt m m v l lt t g t t top po t o mo g t



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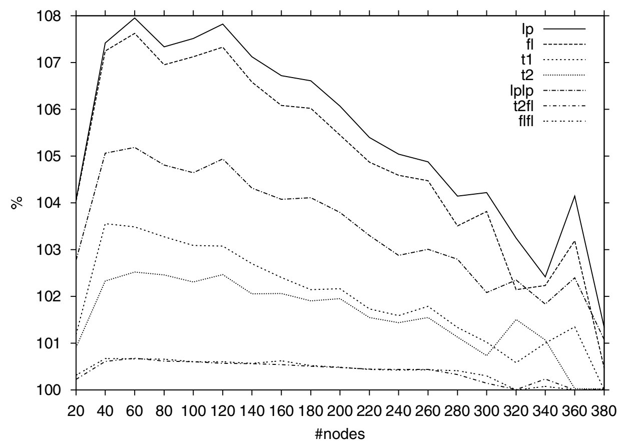
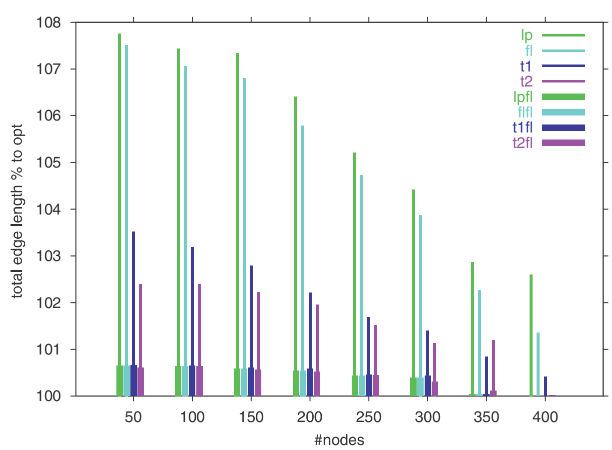
p i t l p i th l p ti l ith



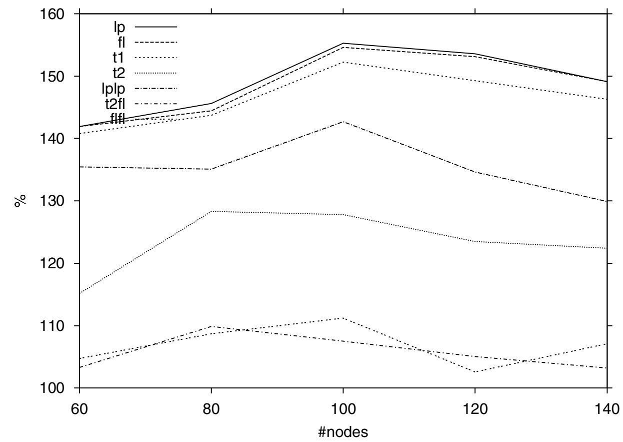
• •• • • • ti l ph t t l l th l ti t pti l lu



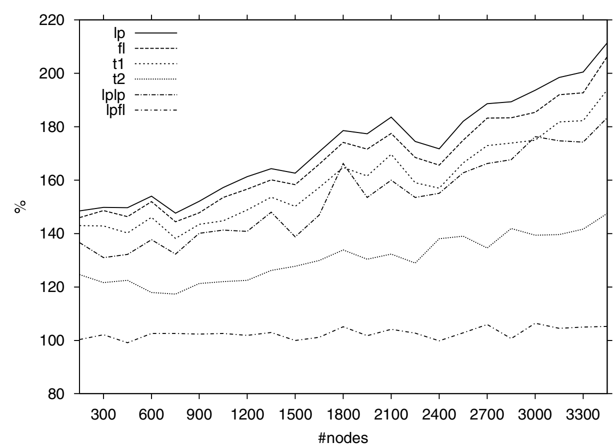
• •• • • • ti l ph i p t th fl th

t m t o . ot o v t o t t t q l t o t m t o l t v l p t o t g p .

l o loo t t t t o t q -t g o - v t g t t q l t o t m t o o 9 gg t t g o to 5 o g l v t . oo T2FL t omp o m t o g. 9. g T1FL FLFL LPFL v lo tog t m g om to t t omp o m t o . t plot pl LPFL t o o t t v t o q l t t t g o . t t t t m t o o t g l t o p o m m- l l t - lo t T2 t t. t g flo omp t o mp ov m t lmo t ll t v t g .



• •• • • • ll qu i-t t t l l th l ti t pti l lu



• •• • • • i qu i-t t t l l th l ti t T2FL

ll mpl m t t o v t o t o t p - t l t t o t t v t t lo .5 o t mp ov m t t t lo . o o ll t . v OPT omp t p ov l opt m l ol t o t lo o o o ll t . t o t omp t o t mo t m t o t v o t m l m t o g p p to v t . t l g t t g t m to p to 75 o o t flo mp ov m t t . ll t t p l p tt o t p o m o t m t o o v ll g p t t LP t v ot mpl m t - t o t m o g t m . o t flo - m t o FL v t g ov T1 T2 lo t mo g t o t t v m t o . o m t m ppl g mp ov m t t .

p i t l p i th l p ti l ith

[g m po t t o o t - g l
o o t 95% t o t g p t . v t m lt t
pt o o t m ll q -t o l o t % o t g l .
o t t m o g t g t t o t o LP
T1 T2. o t o t p t l t t m o t
g o T1 o l o t o t t o t m o LP o T2 o l o t
3 p t o t t m to t to v g l . t
o t o v t m o g to t o T1
to o g l l t m o g o LP l o t m o T2
l t v to LP. v o m ll m o t g t o g
to t t T2 v t o g mp t o t omp t o lt.
m m l g l gt t t omp t g v
m l to t tot l g l gt o t g v g p t . o t o t
p t l g p t v g m m l g l gt p g o p l lo
tog t . mo g t o t t v m t o T1 T2 v l g t v t g
ov t ot m t o . o t tot l g l gt t flo mp ov m t
o l m ll mp t o t lt . lt o t q -t o v o
gg v t o . t T2 p o m t mo g t o t t v m t o
t om t o t t flo mp ov m t t ll t l gt o
ll o t t v m t o . too t flo mp ov m t om t t q l t
o t lt t t lt o t t l m t o v .
p m t lt o t m v o o t flo
mp ov m t lp to t q l t to lo t g p t
t t l m t o ot t o t t v t l lo tog t
o o l l g t mp ov m t flo ppl o t t l l o t.

• • • • •• •• • • •

v v l t t t t -o -t - t omp t o t q o o t ogo l
g p g t m o q l t t m p o m . p opo to
v t t m t o to o t t v t mp ov m t -
t l m t omp t o t q . v omp
t lt o t t ot g t t opt m l g g t
ot . o p m t v t t t t- t to t t t
lgo t m p t l t om to t ollo g m
o l o

p ll o t t om t p t l t o l o v
ll t v o ot t m o q l t g t m . t to
mp o v t t om t l to g t opt m l g
q l t mo mpo t t t g t m . o ov t mpl -
m t t o o t ILP- lgo t m t l to omp t t .
fl

lt o g t lt o t o t t v m t o g tl t -

. . l u, . l i , . ut l

v g flo - omp t o mp ov m t t p. t o p opo to mpl o t t v m t o t g l - t o lo g t p t - oo t g m t ollo flo - po t-p o g t p l l o t t g t m mo g t goo t . mpl m t t o l o t o t v t lo omp to t l t v l ompl t - g l t - t q .

t topolog - p -m t pp o o g t t t l v - t . mo o g g o t t mpl t p o t . o t g pl g p t p t l t t- t o v t t m lmo t ll v t g l p . g p p ov t o t omp t o p o l m.

t o o g omp o t t t omp t o t q m to mpl p ov t m g m t o mo l o g p g l v t g o l g m o g - p . t o t to mo lt t p t t t t m t m v l. g t mo t 5 omp t o t ll o o - l m o o o mpl m t t o o t l t g g. lp to t t t o- m o l omp t o p o l m g p g mo pl o o v t t t t g l- go t m o ll l g m o g p l g p t olog l t lp lot to p ov goo mpl m t t o .

. . ti i, . lli, . i . l ut l ith t -fl i . *s* *t* , - 2) 3 , .

2. . i , . i tti t , . ti i. ut ti ph i - ilit i . *s* *st* , -) , .

3. . ü . ut l. i u pl u ph i i ti l l ut t l . *t* *ss* ,) 33 , .

. . ut l . i i h . pti i i ll i t i l i pl ph. . . u l , . . u , . . i , it , *t* *t* *t* *t* , lu , p 3 3 . p i - l , .

. . ut , . ut l, . i i h . ti i t pl ph. h i l p t, h i h i it¨t i , 2 . u itt pu li- ti .

. . ¨ i . u . i hi h ph ith l u - . . . u , it , , lu 2 , p 2 2 . p i l , .

. . t l i, . i tti t , . i i . puti th l i ith th i i u u . *t* *s* *t* *s* *t* *t* *t* , lu 2 2 , p 33 3 , .

p i t l p i th l p ti l ith

. . i l p , . ¨ i , . u . th l ph i ith t i t . *t s s t t s* . - , 2 .
. . . l u . ut l. u i- th l i pl ph . h i l p t - - - - 3, l titut ü ti , ü , .
. . i , . i tti t , . i i , . i tt , . i , . i- . u - ul it pti l i th l p t ti . *t t t t s* . pp .
. . . l u . ut l. pti l p ti th l i i . . . u l , . . u , . . i , it , *t t t t* , lu , p 3 3 . p i l , .
2. . ut , . ü , . . l u, . ut l. ph i l ith i i ith . h i l p t, h i h i it¨t i , 2 .
3. . *s* . - l - titut ü , i it¨t ll , i it¨t ¨l , . http://www.mpi-sb.mpg.de/AGD.
. ph i t l it t- i t li h li i ph . http://www.dia.uniroma3.it/~gdt.
. . l . i . l ith i p tt th l ph i . . hit i , it , , lu *t t s t* , p 3 2. p i - l , .
. . i tti t , . , . i tt , . i , . i i, . iu. p i t l p i u ph i l ith . *t t t t s*, 3 3 3 , .
. . i tti t , . , . i tt . p i t l p i th ph i l ith . *s t t s t t t* , p 3 3 , .
. . ü . ut l. t h u i ti l ith 2-l t i htli i i i i ti . . . u , it , *s t t t s* , lu 2 , p 33 3 . p i l , .
. . . u , . i lt, . h . p i t l p i - i t i ph i l ith . . . - u , it , *s t t t s* , lu 2 , p . p i l , .
2 . . i . i ph i th i ith th i i u u . *t* , 3) 2 , .
2 . . t i i. th pl it th l p ti . . h , . upt , .- . , . i , it , *t t t s t s t t t s* , lu 3 , p . p i l , .
22. . u . *t t s t t t t.* h il , , .
23. . . l u, . l i , . ut l. p i t l p i th l p ti l ith . h i l p t - - - - 3, h i h i it¨t i , 2 . li i t http://www.ads.tuwien.ac.at/publications/TR/TR-186-1-00-03.

GraphXML — An XML-Based Graph Description Format

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Abstract. GraphXML is a graph description language in XML that can be used as an interchange format for graph drawing and visualization packages. The generality and rich features of XML make it possible to define an interchange format that not only supports the pure, mathematical description of a graph, but also the needs of information visualization applications that use graph–based data structures.

1 Introduction

Most graph drawing systems could benefit from a textual description language to input and, possibly, to output graphs in a human readable form. Ideally, such a description language could also serve as a standardized format for data exchange between systems, enabling information exchange. Defining such a description language is not, by itself, a particularly difficult task: after all, a graph is simply a collection of nodes and edges. Several formats have been proposed in the past such as GML[1] and WebDot's DOT format[2], and the formats used by Rigi[3], LEDA[4], and GDS[5]. None of these are universally supported and they are usually bound to specific systems.

An important development of recent years, which also influences the choice of interchange formats, is the synergy between graph drawing techniques and information visualization. Information visualization has become a well-known research area, with important industrial and scientific applications. Graph drawing techniques play a prominent role in information visualization because the data structures to be visualized can often be described as graphs. The demands of information visualization pose new challenges with respect to graph description formats. It is necessary to include features in a graph description language that are not directly relevant to pure graph drawing. For example, the graph description should be able to include application dependent data[1], either embedded in the description itself or externally referenced (e.g., network statistics for a Web visualizer, genetic data for consensus trees as used in evolutionary research, database references for the result of a database search, etc.). Furthermore, it should be possible to describe composite structures such as nested

[1] Although GML, for example, is a capable description language for graph drawing purposes and includes provisions for extension, the mechanism for associating external data with a graph element is not well defined.

J. Marks (Ed.): GD 2000, LNCS 1984, pp. 52-62, 2001.

hierarchies and clusters. Another less obvious feature is to support the description of the evolution of graphs in time for applications such as animation.

As part of a larger project in graph visualization, we needed a graph description language. Although, initially, we looked for an existing standard, like the ones cited above, none could fully support the needs of information visualization. Consequently, we decided to develop our own format with particular attention to the requirements of information visualization. The result is GraphXML, a graph description language based on XML. Description of this language is the subject of this paper. We hope that this format will enter widespread use. In our view, this would be beneficial for the graph visualization community.

2 Why XML?

XML is a language specification developed by the World Wide Web Consortium that has received significant attention in the last few years[2]. The future evolution of the Web, both in the traditional Internet domain and in mobile communications, is based on XML. There are several reasons for choosing XML as the basis for a graph description format:

- XML defines clear syntactic rules for specifying a “language” for a particular application. An application-specific language is defined in a file called a Document Type Definition (DTD). The end–user also has the option of adding extensions to the language specified in the DTD.
- XML is used by many different applications to define data formats including those for databases, chemical compound definitions, e-commerce, mobile devices, and schematic graphics. A graph interchange format based on XML has a greater chance of being accepted by other application communities.
- There are a number of XML–based specifications that are being defined by communities, both within and outside the World Wide Web Consortium. GraphXML can take advantage of some of these existing standards. See Section 0 for examples.
- Software tools are emerging, which are either based on XML or work with XML. For example, there are a number of both commercial and public domain XML editors. These tools can be very helpful in managing graph description files that are based on GraphXML.
- Several XML parser tools are freely available in Java, C, and C++. A full-featured parser that provides error management and syntax checking can easily be generated using these tools. The main task is to define the semantic interpretation specific to the application (in our case, GraphXML).

[2] There are hundreds of books on XML, so rather than pick a specific reference we refer to the original specification[6], which is available on-line.

In what follows, an overview of the main features of GraphXML is given. A more complete description of GraphXML, including the exact specification, is available in [7].

3 Graph Structures in GraphXML

3.1 A Simple Example

The following code segment shows the simplest possible use of GraphXML. It describes a graph with two nodes and a simple edge:

```
1    <?xml version="1.0"?>
2    <!DOCTYPE GraphXML SYSTEM "file:GraphXML.dtd">
3    <GraphXML>
4      <graph>
5        <node name="first"/>
6        <node name="second"/>
7        <edge source="first" target="second"/>
8      </graph>
9    </GraphXML>
```

This example shows the basic style of a graph description in GraphXML. It resembles the way HTML documents are written, albeit using different tags. This example contains all the elements that are necessary to describe a purely mathematical graph.

The first line is required in all XML files. The second line specifies that this is an XML application based on the GraphXML DTD contained in the file `GraphXML.dtd`[1]. Finally, the third and the last lines enclose the real content of the files. The real content begins with line number 4, which defines a full graph. We delineate graph definitions with the `<graph>` tag so that a file can contain several graph definitions. The body of the graph description is straightforward: two nodes and a connecting edge are defined.

Attributes can also be defined for each of the elements: these are key–value pairs. The GraphXML DTD defines the set of allowable attributes for each element and a validating parser will to check those. It is partly through those attributes that additional information about nodes, edges, or graphs can be conveyed to the application. For example, the `<graph>` tag can use keys such as `version`, `vendor`, `preferredLayout`, `isPlanar`, `isDirected`, `isAcyclic`, or `isForest`.

3.2 Application-Dependent Data

Application data can be added to different levels of the graph description through a series of additional elements. These elements are meant to represent the different types of application or domain-dependent data that can be associated with a graph, a node, or an edge. GraphXML defines the following application data:

[1] This line specifies that the DTD is on the local file system but could be replaced by a URL specification. In this way, it is possible to make the DTD publicly available on the World Wide Web.

- **Labels** (<label> tag): whereas the name attribute is used for unique identification, the label can contain any kind of text and does not have to be unique. Applications can use these to label nodes and edges.
- **Data** (<data> tag): domain-dependent data represented by a node, edge, or even the full graph[1].
- **Data references** (<dataref> tag): data that is referenced externally rather than embedded in the graph description file.

The format of external references (within the <dataref> tag) follows the specification of a separate document of the World Wide Web consortium, called XLink[8]. For our purposes, the virtually identical URL format used in HTML will suffice for external references.

The following example describes the same graph as in the previous section, except that the first node has associated data[2].

```
1          <node name="first">
2           <label>Project Home page</label>
3            <data> CWI Information Visualization project</data>
4            <dataref>
5             <ref xlink:role="Lead"  xlink:href="http://…/~ivan"/>
6             <refxlink:role="Descr"xlink:href="http://…/InfoVisu"/>
7            </dataref>
8           </node>
9          <node name="second"/>
10         <edge source="first" target="second"/>
```

Note the use of the xlink:role attribute in the example (see lines 0 and 0), which can be used to describe what the exact role of the link is. This can provide a useful indication to the application.

3.3 Hierarchical Graphs

All the examples mentioned until now refer to a single graph. However, information visualization applications are often used on very large graphs, and one of the ways of handling the size problem is to define the information in terms of a hierarchy of graphs, or clusters, instead of one single graph. What this means is that the nodes of one graph can refer to other graphs, these can refer to yet another graph, and so on. Powerful techniques exist to hierarchically cluster graphs and to visualize the hierarchies (see the survey of Herman *et. al.*[9]). GraphXML offers a way to describe such graph hierarchies. Consider the following example:

```
<graph id="L-1">
  <node name="first"/>
  <node name="second"/>
```

[1] Although XML describes everything in terms of strings, it has its own formalism, called entities and notations, which can be used to include binary data, too. However, using external references, through the <dataref> tag, may be more appropriate for this.

[2] From now on, we are omitting the header part from the examples to save space.

```
4            <edge source="first" target="second"/>
5          </graph>
6          <graph id="L-2">
7            <node name="third"/>
8            <node name="fourth"/>
9            <edge source="third " target="fourth"/>
10         </graph>
11         <graph id="levelTwo">
12         <node isMetanode="true" name="cluster1" xlink:href="#L-1"/>
13         <node isMetanode="true" name="cluster2" xlink:href="#L-2"/>
14         <edge source="cluster1" target="cluster2"/>
15         </graph>
```

The example shows the tools introduced in GraphXML to describe graph hierarchies. A GraphXML document can contain several graph descriptions. Each graph description can use a (unique) identifier, using the `id` key. A "meta" node in a higher level in the hierarchy uses this identifier to "`link`" to another graph, using the `xlink:href` attribute key (see line numbers 0 and 0). The `isMetanode` attribute is used to unambiguously identify a node that refers to another graph. Using these definitions, this example describes a two level graph with two nodes and one connecting edge, where each node represents another graph. The full format of the `xlink:href` value is: `URL#identifier` where the identifier refers to a graph identifier *within* the document referred to by the URL. If the target is in the same document as the source, the URL part can be left out (as in the example).

This simple adaptation of HTML introduces an powerful feature to GraphXML. It is possible to define a hierarchy of graphs, consisting of graphs that that are located in another file, or possibly in another Internet location than the hierarchy description itself. Applications that make use of this capability can create their own hierarchical or clustered views of public datasets.[1]

3.4 Dynamic Graphs

If a graph visualization system is used interactively, the system may be asked to store the history of the user's actions in some form of journaling. What this means is that an interchange format should be able to describe not only the initial graph, but also any editing steps that have changed the structure or the attributes of the graph. This is the reason for the use of `<edit>` tags in GraphXML.

The edit sections in a GraphXML document are syntactically similar to graph specification, except for the use of the `<edit>` tag instead of `<graph>`. Furthermore, the `<edit>` tag has a required attribute key `action`, whose value can be `remove` or `replace`. Here is an example:

[1] Note that WebDot's DOT format[2] includes facilities to describe clusters, but the format is limited to subgraphs defined in the same file.

```
1          <graph version="1.0" vendor="cwi" id="theGraph">
2            <node name="first">
3              <label>A label to display for this node</label>
4              <dataref> <ref xlink:href="BigIcon.gif"/> </dataref>
5            </node>
6            <node name="second"/>
7            <edge name="thisEdge" source="first" target="second">
8          <dataref> <ref xlink:href="ExternalData.bmp"/> </dataref>
9            </edge>
10         </graph>
11         <edit action="replace" xlink:href="#theGraph">
12           <node name="first">
13           <label>Another label</label>
14           <dataref> <ref xlink:href="anotherImage.gif"/> </dataref>
15           </node>
16           <edge name="thisEdge" source="first" target="second"/>
17         </edit>
```

The semantics of the editing element (line 0) is based on identifying the element in the edit block and in the original graph. This controls what is being edited. The value of the `action` attribute determines what happens: the corresponding element will either be removed or replaced by the content in the `<edit>`. The semantics of matching elements is more involved (see the full description[7] for details).

The result of carrying out the editing action in the above example can be represented by the following GraphXML description:

```
1          <graph version="1.0" vendor="cwi" id="theGraph">
2           <node name="first">
3            <label> Another label </label>
4           <dataref> <link xlink:href="anotherImage.gif"/> </dataref>
5            </node>
6            <node name="second"/>
7            <edge name="thisEdge" source="first" target="second"/>
8          </graph>
```

Note the disappearance of the data references in the edge (line 0 in the previous example). This is because the editing action has replaced those with their "empty" counterpart from within the editing element (line 0 in the previous example).

If, in the same example, the `action` attribute were set to `remove`, the result would be as follows:

```
1  <graph version="1.0" vendor="cwi" id="theGraph">
2   <node name="first"> </node>
3   <node name="second"/>
4   <edge name="thisEdge" source="first" target="second">
5   </edge>
6  </graph>
```

The `xlink:href` attribute used by the edit element has the same syntax and semantics as described for hierarchical graph descriptions. In other words, an edit element can refer to a graph in another file or even another Internet location.

As an additional tool for editing, GraphXML also defines the `<edit-bundle>` tag, which is simply a sequence of edit tags:

```
1 <edit-bundle>
2  <edit …> … </edit>
3  <edit …> … </edit>
4 </edit-bundle>
```

This simple grouping of editing elements can be useful if the user wants to animate the result of editing, but doesn't want to display each individual editing step. Using this bundling mechanism, the granularity of animation can be controlled by the creator of the GraphXML file.

4 Storing Geometry

Section 0 describes only structural elements: nodes, edges, and hierarchies. Visualization systems have to layout the graph before presenting it to the user. GraphXML tags for this purpose are described in the next section.

4.1 Positions and Size

The position of a node can be described by adding the `<position>` tag as a child to `<node>`:

```
<position x="0.0" y="0.0"/>
```

The size of the node can also be described with the `<size>` tag:

```
<size width="3.0" height="5.0"/>
```

This tag can be especially important for layout algorithms that take the node size into account when laying out the graph.

The `<size>` tag can also be used as a direct child of `<graph>`. It then denotes the size, or bounding box, of the full graph. Applications can benefit from such information because it allows them to allocate the necessary area on the screen and coordinate transformations in advance.

4.2 Edge Geometry

Edges differ from nodes insofar as a sequence of coordinates may be necessary. This is achieved through the `<path>` tag:

```
<path type="polyline">
   <position x="0.0" y="0.0"/>
   <position x="0.1" y="0.0"/>
   <position x="0.1" y="0.1"/>
</path>
```

The `<path>` tag contains a sequence of control points. The `type` attribute can take the value of `polyline`, `arc`, or `spline`, depending on whether the edge is to be drawn as a polyline or a spline curve. In the case of a spline, the positions indicate the spline control points.

4.3 Geometry for Hierarchical Graphs

The geometry definition described earlier is insufficient for the description of hierarchical graphs: the same graph might be included in more than one place in the higher-level graph and the geometry must be adapted to the metanode's position. The solution is to use the `<transform>` tag, which is a child of `<node>`. This element describes the transformation to be applied to each coordinate value in the referenced graph. For example, the following code fragment:

```
<node name="SecondOrder" isMetanode="true" xlink:href="#basic">
    <transform matrix="1.0 0.0 0.5 0.0 1.0 0.5"/>
</node>
```

translates all the referred nodes and points to the (0.5,0.5) point. The `<transform>` element contains 6 numbers to describe a 2×3 matrix. See [10] for details on how these transformation matrices are used in computer graphics.[1]

5 Visual Properties

In addition to layout, the appearance of a graph is determined by visual properties, such as line width, colour of the components, icons replacing nodes, etc. In GraphXML, the user can control these properties through the `<style>` tag. A style can include the tags `<line>` or `<fill>`. In the case of a node, the line tag controls the border of the symbol drawn for the node, whereas the fill tag controls the interior. For example, the block:

```
1 <node name="first">
2  <style>
3    <line linestyle="dashed" linewidth="2" colour="red"/>
4    <fill fillstyle="solid" colour="blue"/>
5  </style>
6 </node>
7 <edge source="first" target="second">
8   <style>
9     <line linestyle="solid" linewidth="1" colour="cyan"/>
10    <fill fillstyle="none"/>
11   </style>
12 </edge>
```

defines a node symbol to have a red dashed boundary drawn with a line width of 2, and filled with solid blue[2]. The edge should be drawn in cyan without filling the interiors (in case the edge is drawn as a polygon).

The fill element can also refer to an image file instead of specifying the colour and the fill style. This instructs the application to use the image as an icon to display the node. For example, line 0 could be replaced by:

```
      <fill xlink:href="http://www.some.site/imagefile.gif"/>
```

[1] All positions, as well as the transformations, can also be extended to 3D.

[2] Note that this specification does not specify the exact glyph to be drawn by the application. This is either left to the implementer of the visualization system, or specified via a more sophisticated control tag, called `<implementation>`. See [7] for further details.

The mechanism described so far would lead to repeated visual control tags, greatly increasing the size of the graph file. It might also become cumbersome to adapt a graph file to a new environment with other visual characteristics. To solve these problems, an inheritance mechanism for visual properties is available. The goal is to provide a general control mechanism that allows for easy adaptation. A style element can be added on the graph level, to control the overall appearance of the graph:

```
<graph>
 <style>
  <line tag="node" linestyle="dashed" linewidth="2" colour="red"/>
  <fill tag="node" fillstyle="solid" colour="blue"/>
  <line tag="edge" linestyle="solid" linewidth="1" colour="cyan"/>
  <fill tag="edge" fillstyle="none"/>
 </style>
 …
```

This results in the same visual effect as before, except that the visual properties are valid for *all* nodes and edges in the graph (note the use of the `tag` attribute in the line and fill elements to differentiate between nodes and edges).

Nodes and edges can also use the `class` attribute to categorize elements with common visual properties. Using this additional identification, finer control over the visual attributes can be achieved by applying a specific visual attribute to a class of nodes or edges. For example, in following block of code:

```
1 <graph>
2  <style>
3   <line tag="node" linestyle="dashed" linewidth="2" colour="red"/>
4   <fill tag="node" fillstyle="solid" colour="blue"/>
5   <fill tag="node" colour="green" class="special"/>
6  </style>
7  …
8  <node name="first"/>
9  <node name="second" class="special"/>
10  …
11  <node name="nth" class="special"/>
12  …
```

the node `first` will be displayed the same way as before; however the nodes `second` and `nth` will become green instead of red. This is because line 0 specifies that “all nodes of class ‘special’ should be filled in green”.

Metanodes require a special style control facility to affect the style of all included elements. The reader should refer to [7] for details.

Using the entity mechanism of XML, it is possible to include an XML file within another. This is particularly handy when controlling styles: it is possible to collect all the style elements into a separate file and include it in a graph specification. If a change is made to the style file, it will automatically affect the visual properties of all the graph description files that reference it.

6 User Extensions

Although the specification of GraphXML includes rich facilities for the association of data with nodes and edges, it is not possible to predict all the possible attributes an application might want to add to an element. For example, if the application is for web visualization, the user might want to associate a MIME type with a node. In other

words, the application would need to have its own, `<mime>` tag for each node, in addition to the tags defined by the GraphXML DTD.

Such extensions are possible using GraphXML. This is how an extension for a mime tag can be added to a graph:

```
1 <!DOCTYPE GraphXML SYSTEM "file:GraphXML.dtd" [
2 <!ENTITY % nodeExtensions "|mime">
3 <!ELEMENT mime EMPTY>
4 <!ATTLIST mime
5    type        CDATA #REQUIRED
6    application CDATA #IMPLIED
7 >
8 ]>
9 <GraphXML>
10   <graph>
11    <node name="first">
12      <label>Project Home page</label>
13      <dataref> <ref xlink:href="Description.pdf"/> </dataref>
14      <mime type="application/pdf" application="Adobe Acrobat"/>
15    </node>
16    …
```

The file contains what is called an "internal DTD" (lines 0–0), which extends the elements that can be included in a node definition. The definition states that a `<mime>` element can be added as the child of a node, that this is an element that contains only attributes (i.e. no sub–elements), and that the attributes can be `type` and `application` (the first being compulsory, the second optional).

The XML syntax is a bit cryptic. However, the end–user does not necessarily have to include such internal DTD's into all GraphXML files. Instead, using standard XML mechanisms, it is possible to define a *separate* DTD containing the application-specific extensions (and a reference to the `GraphXML.dtd`, of course). Using such extension, the header part becomes simply:

```
<!DOCTYPE GraphXML SYSTEM "file: WebVisualizer.dtd ">
<GraphXML>
 <graph>
   …
```

In other words, the intricacies of the XML DTD syntax remain invisible to most users. Furthermore, because the extension is made through a standard XML mechanism, the basic GraphXML parser remains unchanged. Only the application–dependent part has to be adapted for the new extensions. Herman and Marshall [7] describes application-specific DTD's in more detail.

7 Future Developments

As stated REFearlier, one of the advantages of using XML is that the future evolution of XML-based specifications can be used to develop new tools for GraphXML. Here are just two examples:

- An upcoming specification is the RDF Schemas[11], which will replace DTD's in future. Schemas also include various data types with possible constraints on the values. Schema based parsers will be able to make on-the-fly checks on the input

data (e.g., coordinates should be numbers), relieving the GraphXML parser and applications from having to perform these checks themselves.

- W3C is currently developing a standard for graphics on the Web, called SVG[12]. It will be possible to define simple tools to transform GraphXML specifications into SVG[1]. This means that the result of graph drawing systems can be published on the Web in the form of vector based graphics, rather than screen dumps, yielding better quality and smaller bandwidth requirements.

8 Public Availability

The full description of GraphXML, as well as the GraphXML DTD, is publicly available at the URL: http://www.cwi.nl/InfoVisu/GraphXML. The parser is also available as a collection of Java 1.2 classes, which can be embedded into an application. The implementation uses publicly available XML parsers in Java. See the full description at the above URL for details.

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[1] Standard XML-based tools can perform such a transformation. Also, a graph vizualization application that reads GraphXML and saves to SVG format can be found at http://www.cwi.nl/InfoVisu/GVF/ .

On Polar Visibility Representations of Graphs

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Abstract. We introduce polar visibility graphs, graphs whose vertices can be represented by arcs of concentric circles with adjacency determined by radial visibility including visibility through the origin. These graphs are more general than the well-studied bar-visibility graphs and are characterized here, when arcs are proper subsets of circles, as the graphs that embed on the plane with all but at most one cut-vertex on a common face or on the projective plane with all cut-vertices on a common face. We also characterize the graphs representable using full circles and arcs.

1 Introduction

Visibility graphs are now a well-established area of graph drawing [10]. Much has been written about their importance and application; however, they continue to pique the imagination of mathematicians with their intrinsic appeal and intriguing questions [2]. There has been a natural progression from bar-visibility graphs (BVGs) [11, 17] to rectangles [1, 3, 4, 6], from bars with visibilities in the plane to those on the sphere and cylinder [12, 13], on a (flat) torus [8], or on the Möbius band [5]. These rectilinear representations are natural ones for most applications; however, we turn instead to the realm of polar representations with arcs of circles and radial visibility. In many ways circular representations and related polar coordinates are equally natural and in some contexts more applicable than rectilinear ones. With this change of perspective we can and do represent a class of graphs, larger than with bars in the plane, though ultimately constrained by the real projective plane. Thus with a planar representation of arcs of circles nonplanar graphs are drawn in a natural way, resulting in diagrams often reminiscent of time-exposed shots of the North Star and surrounding stars.

We introduce the layout of graphs as polar visibility graphs (PVGs) using arcs of concentric circles (arcs that are proper subsets of a circle) with radial visibility, including visibility through the origin, the center of all the concentric circles. These graphs, though arising naturally from visibility in the plane, corres-

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pond to graphs embedded on the (real) projective plane, the nonorientable surface of Euler characteristic 1. PVGs are characterized as the planar graphs that can be drawn in the plane with all but at most one cut-vertex on a common face plus the graphs that can be embedded on the projective plane with all cut-vertices on a common face. We also consider the variation in which full circles are allowed along with arcs, and characterize the graphs so representable (CVGs) in terms of their block-cutpoint tree.

2 Background

Just as visibility wider than along a line is required for BVGs, we ask that radial visibility in PVGs be available through a nondegenerate cone. Define a (nondegenerate) *cone* in the plane to be a 4-sided region of positive area with two opposite sides being arcs of circles, centered at the origin, and the other two sides, possibly intersecting, being radial line segments on lines through the origin. Thus, both $\{(r, \theta): 1 \le r \le 2,\ 0 \le \theta \le \pi/6\}$ and also

$$\{(r,\theta): 0 \le r \le 1,\ 0 \le \theta \le \tfrac{\pi}{6} \text{ or } \pi \le \theta \le \tfrac{7\pi}{6}\} = \{(r, \theta): -1 \le r \le 1,\ 0 \le \theta \le \tfrac{\pi}{6}\}$$

are considered to be cones, respectively, not containing and containing the origin. Given a set of arcs, all centered at the origin, two of these arcs a_1 and a_2 are said to be *radially visible* if there is a cone that intersects only these two arcs and whose two circular ends are subsets of the two arcs; the same definition holds for visibility between an arc and a circle and between two circles. A graph is called a *polar visibility graph* if its vertices can be represented by arcs, including endpoints, of circles centered at the origin, having pairwise disjoint relative interiors, so that two vertices are adjacent if and only if the corresponding arcs are radially visible; see Figure 1a below for a PVG representation of K_6. If this model is used, but without visibility through the origin, the graphs arising are one of the cylindrical types characterized in [13]. Note that for a 2-connected graph (a graph without cut-vertices) there is no loss in taking arcs as proper subsets of circles

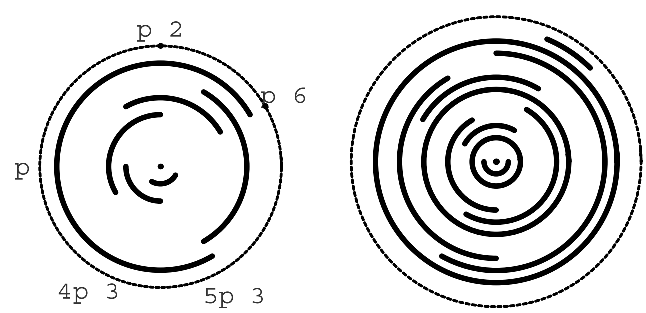

Fig. 1a. K_6 **1b.** A circular visibility layout

since a full circle can be cut down to a smaller arc, leaving the same visibilities. Arcs in a PVG layout spanning more than half its circle will provide interesting variations, full circles even more. We use graph theoretic terminology as in [15], topological notions as in [9], and algorithmic ideas following the BVG presentation in [10].

Similarly a graph is called a *circular visibility graph* if its vertices can be represented by arcs and circles with radial visibility between arcs and circles determining edges as for PVGs. When possible we prefer, but do not require, arcs over circles; that is, in a layout we will decrease a circle to become a proper arc if no additional visibilities are introduced. We shall see that some planar and projective planar graphs with cut-vertices on an arbitrary number of faces are CVGs, but not PVGs, but that these faces must be nested appropriately. Figure 1b shows such a planar CVG. In that layout the inner circle contains one arc; if instead, it contained four mutually visible arcs, encircling the origin and forming a K_5, the example becomes a nonplanar CVG.

Note that in a PVG or CVG layout of a graph G, we may draw each arc and circle on a distinct circle, and we may take these circles to have radii $1, 2, \ldots, n$ where $n = |V(G)|$. This naturally leads to another layout of the graph in a disk of radius n+1 and centered at the origin by inverting each circle and arc through the circle of radius $(n+1)/2$. That is, each point with polar coordinates (r, θ), $0 < r < n+1$, is mapped by the inversion to the point $(n+1-r, \theta)$. This inversion preserves circles, arcs, and the angles defining these arcs. If the original layout was L, we denote this inverted layout by $I(L)$.

Recall that the (real) projective plane can be obtained by taking a circular disk and identifying opposite (or antipodal) points. Thus if we identify opposite points of the circle of radius $n+1$, we create a projective plane. Two arcs in $I(L)$ (or an arc and a circle or two circles) that were previously radially visible in a cone, not containing the origin, are still radially visible, and a pair visible in a cone through the origin are now visible in a "generalized cone" that crosses the boundary of the projective plane, reemerging on the other side. The coordinates of such a generalized cone are given by $\{(r,\theta),\ r^{\star} \le r \le n+1 \text{ or } -(n+1) \le r \le -s^{\star},\ \theta_1 \le \theta \le \theta_2\}$ where $r^{\star}$, s*, $\theta_1 < \theta_2$ are constants, $0 \le r^{\star},\ s^{\star} < n+1$. In addition, the interior of no two of these new cones intersect. Fig. 2a shows the inverted layout of K_6 on the projective plane with dashed lines indicating a conical area of visibility, and in 2b we see an embedding of K_6 created by shrinking each arc to a vertex. The first proposition is then clear since each inverted arc and circle on the projective plane can be replaced by a single vertex. Then the visibility cones can each be shrunk and transformed to a set of nonintersecting edges on the projective plane.

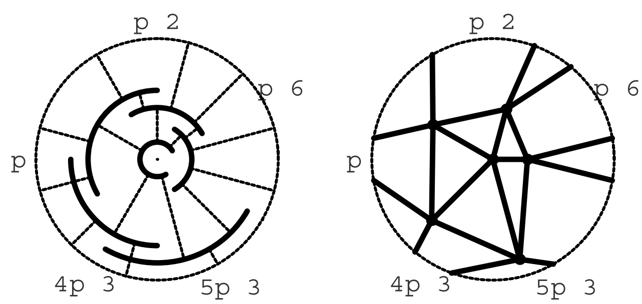

Fig. 2a. $I(K_6)$ and $I(K_6)^*$ on the projective plane
2b. For $G = K_6$, $I(L_G) = (I(L))_G$

Proposition 2.1. A PVG or a CVG embeds on the projective plane.

Recall that a graph G is said to *embed* on a surface S if it can be drawn there without any edge crossings, and that each maximal connected component of $S \setminus (V(G) \cup E(G))$ is called a *face* of the embedding (we do not require that the faces be simply connected).

Theorem 2.2. A graph G is a PVG if and only if either a) G has an embedding in the plane with all but at most one cut-vertex on a common face, or else b) G has an embedding on the projective plane with all cut-vertices on a common face.

Note that condition (a) allows for the representation of planar graphs that are not BVGs; for example $K_{2,3}$ with three additional vertices of degree 1 appended, one each to a vertex of degree two, is a PVG. Similarly $K_4 + 4e$ (K_4 plus a pendant vertex and edge at each vertex) is a PVG (see Fig. 3); these are the smallest graphs that are not BVGs. Condition (b) also allows for more planar graphs; for example, two vertices joined by three internally disjoint paths of length three (i.e., three edges each) plus six vertices of degree 1, each adjacent to a different vertex of degree two, satisfies (b), but not (a).

Every graph G can be decomposed into its blocks and their connecting cut-vertices (a *block* is either an edge or a 2-connected subgraph; see [15]), and these connections determine a tree, called the *block-cutpoint tree* of the graph, BC(G). This tree has a vertex for each block and for each cut-vertex of G, and two vertices of BC(G) are adjacent if and only if they correspond to an incident cut-vertex and block. We call a block *planar* if it represents a planar graph.

Theorem 2.3. A graph G is a CVG if and only if BC(G) consists of a path $P = (e_1, e_2, \ldots, e_{2k+1})$, $k \geq 0$, with e_{2i} representing a planar block, $i = 1, \ldots, k$, so that

1a) e_1 is also incident with one additional (nonempty) block representing a (2-

connected) projective planar graph, or

1b) e_1 is also incident with one or more (nonempty) planar blocks, and

2) e_{2k+1} is also incident with an arbitrary tree structure T so that $T \cup \{e_{2k+1}\}$ represents a planar graph that can be drawn in the plane with all cut-vertices, except possibly for that representing e_{2k+1}, on a common face.

When $k = 0$, these conditions reduce to those of Theorem 2.2. On the other hand, it may be that each cut-vertex of G, represented by $e_1, e_3, \ldots, e_{2k+1}$, lies on a different face, as in Figure 1b. This example is the first of an infinite family of CVGs with an increasing number of cut-vertices, all on different faces; the family is obtained by nesting repeatedly the same pattern of arcs and circles. Most of the details of the proofs of Theorems 2.2 and 2.3 are included below.

As described in [10], planar layouts and the block-cutpoint tree of a graph can be determined in linear time. Projective planar graphs can also be recognized and embedded in linear time [7]. It can quickly be determined whether all cut-vertices of a graph lie on a common simple cycle and, if so, whether there is an embedding in either surface in which this cycle bounds a face. The proofs of Theorems 2.2 and 2.3, together with standard BVG algorithms, lead to a $O(NE^2) = O(N^3)$ time algorithm for laying out a PVG G with N vertices and E edges, given an embedding of G in the projective plane as a rotation scheme (defined below), as in [7, 9].

3 Main Results on PVGs

We develop theory that will also allow extension to CVGs. We focus on simple graphs and their characterizations as in Theorems 2.2 and 2.3. Thus we say that two arcs are radially visible if there is at least one maximal cone providing mutual visibility; however, we can also obtain more precise results by keeping track of multiple and even self-visibility between arcs and circles.

First we need more precise topological and geometric definitions. Consider a PVG or CVG layout L of a graph G and its inverse layout on the projective plane, $I(L)$. We let L^* (respectively, $I(L)^*$) denote the visibility depiction obtained by shrinking each maximal visibility cone of L (resp., $I(L)$) to a distinct line segment by reducing its angles $\beta_1 \le \theta \le \beta_2$ to some constant $\theta = \beta$, $\beta_1 < \beta < \beta_2$; strict inequality ensures distinct visibility segments. For $G = K_6$, $I(L)^*$ is shown in Fig. 2a.

Also let L_G (resp., $(I(L))_G$) denote the graph obtained from L^* (resp., $I(L)^*$) by shrinking each arc to a vertex, consisting of one point, and transforming each visibility line segment to an edge that intersects no other edge except possibly at the origin (resp., an edge that intersects no other edge on the projective plane). If L^* or $I(L)^*$ contains a circle, it is replaced by a point as vertex. Thus $(I(L))_G$ is a graph embedded on the projective plane, see Fig. 2b. Note that

L^*, $I(L)^*$, $I(L_G)$, and $G(I(L))$ have visibility segments and edges for each distinct, maximal visibility cone so that multiple edges and loops may be present in these depictions; however, a pair of multiple edges will not form an embedded digon with empty interior.

Note that the complement of the arcs, circles, and lines of L^* divide up the plane into faces; similarly $I(L)^*$ divides up the projective plane. One face of L^* is the exterior face, possibly containing the origin; this exterior face is the one in which most cut-vertices of a PVG and their blocks can be placed. We say that an arc or circle of a layout L lies on the exterior face if it lies on the exterior face of L^*.

We use the following combinatorial description of an embedded graph and of a PVG or CVG layout. If a graph is embedded on any surface, then for each vertex there is naturally defined a cyclic rotation of its neighbors, given by the order, say clockwise, of its edges in the embedding; such a collection of rotations, one for each vertex, is called a *rotation scheme*. (See, for example, [16] where it is shown that an embedding is equivalent to a rotation scheme.) Such a description is generally used for algorithms on embedded graphs [9]. Similarly, given a PVG layout L in the plane and its inverse layout $I(L)$ in the projective plane, one can define the *arc-rotation scheme* to be the set of cyclic rotations of neighbors about each arc of its visibilities to other arcs; note that the rotations at the arcs of L and of $I(L)$ are inverses of each other. We say that an embedding of a PVG graph G in the plane or on the projective plane and its polar visibility layout L are *equivalent* if the arc-rotation scheme of $I(L)$, when translated into a set of vertex-neighbor cycles, yields the rotation scheme of the embedded graph; see Figs. 1, 2. Given a circle in a CVG layout L or $I(L)$, the neighbors divide into two cyclic rotations of the inner and outer visibilities, called the *circle-rotation scheme*. Then a drawing of a CVG and its layout L are equivalent if the arc/circle-rotation schemes of $I(L)$ agree with those of the embedded graph.

It is not hard to see the following, by bending or straightening corresponding BVG and PVG layouts.

Proposition 3.1. A connected graph has a PVG layout with no visibilities through the origin if and only if the graph is a BVG.

A PVG layout with no visibilities through the origin contains arcs in sectors, alternating about the origin, so that some can be reflected through the origin, leaving the layout in two quadrants, and this can be straightened to form a BVG.

Of course there are planar graphs with layouts as PVGs including visibilities through the origin and with cut-vertices represented on the exterior face. Note that whenever there are visibilities through the origin in a layout L, then the equivalent graph $G(I(L))$ is embedded on the projective plane. It turns out that in some PVG layouts there is a (sneaky) hiding place for a cut-vertex and its connecting blocks, but the resulting graphs turn out to be planar. In a PVG

layout we call an arc $a^\star$ a *long arc* if its angular span is greater than π. Suppose $a^\star = \{(r^\star, \theta), 0 \le \theta \le \pi + \xi\}$, for some $0 < \xi < \pi$. Then the cone defined by $C(a^\star) = \{(r, \theta), -r^\star \le r \le r^\star, 0 \le \theta \le \xi\}$ is an area in which interior arcs can see the arc $a^\star$ and possibly no others; see Fig. 3.

In preparation for CVG layouts, we require special PVG layouts. Let $a^\star$ be a long-arc at radius 1, spanning $\theta_1 \le \theta \le \theta_1 + \pi + \xi$ for some $\xi > 0$. Arcs $a^\star$ and $b^\star$ are called a *long-arc pair at the origin* if they are mutually visible, together they span 2π, and if $b^\star$ lies at radius $r^\star > 1$, no arcs intersect the *long-arc cone* $\{(r, \theta): 0 \le r < r^\star, \theta_1 + \pi + \xi < \theta < \theta_1 + 2\pi\}$. (For example, when $r^\star = 2$, no arcs can meet the designated cone.) Similarly if $a^\star$ is a long-arc at the outermost radius $n = |V(G)|$, spanning $\theta_2 \le \theta \le \theta_2 + \pi + \psi$ for some $\psi > 0$, then $a^\star$ and $b^\star$ are a *long-arc pair at infinity* if they are mutually visible, together span 2π, and if $b^\star$ lies at radius $r^\star < n$, no arcs intersect the long-arc cone $\{(r, \theta): r^\star < r < n, \theta_2 + \pi + \psi < \theta < \theta_2 + 2\pi\}$; see Figs. 1a, 3. Notice that in long-arc pairs the long arc at radius 1 or at radius n could be extended to form a full circle without changing visibilities.

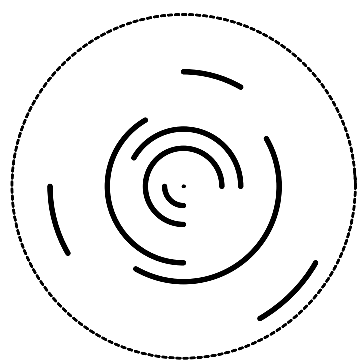

Fig. 3. K_4+4e.

Here are the building-block results needed for the PVG characterization.

Proposition 3.2. Let G be laid out as a PVG L including a long arc $a^\star$ that represents a cut-vertex $x^\star$, not lying on the exterior face, and let B be a block of G incident with $x^\star$ and whose representation lies within $C(a^\star)$ in L. Then G is a planar graph and can be drawn in the plane with one face including all vertices whose arcs lie on the exterior face of L.

The proof consists of observing that in $I(L)$ and in $(I(L))_G$ the representation of a block B incident with $x^\star$ lies within a (noncontractible) sector of the projective plane that divides the space into two contractible (planar) regions.

The next result is the necessary topological argument needed to characterize PVGs; it is a contraction proof similar to that of [14] and [8]. This result is

carried out for multigraphs, those embedded with no digon face with empty interior except for two special faces. For these graphs we achieve layouts with a one-to-one correspondence between distinct, maximal visibility cones and edges of G. If x is a vertex of a PVG, we let a_x denote its arc in the layout, and conversely x_a is the vertex corresponding to an arc a.

Proposition 3.3. (i) Let G be a loopless 2-connected plane multigraph, let F be a face in the embedding, and let c be a vertex of G. Suppose G has at most two digon faces, possibly F and, when c does not lie on F, possibly one incident with c. Then $G^{ç} = G$ plus a loop at c has a PVG layout $L^{ç}$ in which all vertices of F are represented on the exterior face of $L^{ç}$ and (a_c, a_d) is a long-arc pair at the origin for some neighbor d of c. In addition, $G^{ç}$ has an embedding on the projective plane that is equivalent to $L^{ç}$.

(ii) Let G be a loopless 2-connected plane multigraph with v_1 and v_2 designated, distinct vertices and with no digon face. Then $G^{ç} = G$ plus a loop at v_2 has a PVG layout $L^{ç}$ with v_i represented by arc a_i, $i = 1, 2$, with (a_1, b_1) a long-arc pair at infinity, and with (a_2, b_2) a long-arc pair at the origin, where for $i = 1$ and 2, arc b_i corresponds to some neighbor of v_i. Also $G^{ç}$ has an embedding on the projective plane that is equivalent to $L^{ç}$.

(iii) Let G be a loopless 2-connected multigraph with a 2-cell embedding on the projective plane, with F a face in the embedding, and with no digon face except possibly for F. Then G has a PVG layout L that is equivalent to the embedding of G with exterior face corresponding to F.

(iv) Let G be a loopless 2-connected multigraph with a 2-cell embedding on the projective plane, with no digon face, and with v_1 a designated vertex. Then G has a PVG layout L, equivalent to the embedding of G, in which v_1 is represented by arc a_1 with (a_1, b_1) a long-arc pair at infinity and with arc b_1 corresponding to some neighbor of v_1.

Sketch of proof of (i). For most cases (when c does not lie on F), the proof is by induction on n.

We can always find a nonloop, nonmultiple edge $e = (x, y)$ of G so that G with e contracted, G/e, is 2-connected, loopless, embedded on the plane, and F is still bounded by at least two edges. G/e satisfies the inductive hypothesis and so has PVG layout L_e, equivalent to an embedding of G/e plus a loop on the projective plane. If the contraction combines vertices x and y into new vertex x^*, let a^* be its representation in L_e, at say radius r. Because the embeddings of G/e and L_e are equivalent, the lines of visibility to arcs representing vertices adjacent to x in G are consecutive in the rotation of visibility lines about a^* in L_e. Then, when a^* is not one of the special long arcs at the origin, it can be replaced by two arcs a_x and a_y at radii $r - 0.5$ and $r + 0.5$ (or vice versa), representing vertices x

and y of G, so that their visibilities give all edges incident with x and y and preserve the arc-rotations at x and y in G; see Fig 4. This alteration gives the desired PVG layout L for G. The argument is similar, though a bit more intricate, when $a^{\star}$ is part of the long-arc pair at the origin. The proofs for (ii–iv) are analogous.

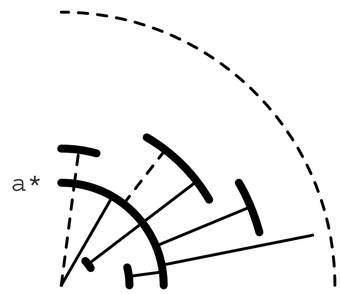

Fig. 4. An arc and its neighbors

We then obtain the following.

Proposition 3.4. If G has a PVG layout L, then the embedding HIHLLL(of G on the projective plane has cut-vertices on at most two faces. If the embedding has cut-vertices on two faces, then on one face there is only one cut-vertex, represented in L by a long arc.

Corollary 3.5. If G has a PVG layout L with a long arc $a^{\star}$, representing a cut-vertex $x^{\star}$ and not lying on the exterior face, then G has a planar embedding with all cut-vertices except for $x^{\star}$ lying on a common face.

Theorem 3.6. A simple planar graph G has a PVG representation if it has a planar embedding with all but at most one cut-vertex on a common face.

Sketch of the proof. Assume that G is not a BVG and so can be drawn in the plane with cut-vertices lying on the exterior face F_1 and an additional cut-vertex c lying on $F_2 \neq F_1$. Consider the block-cutpoint tree $BC(G)$ of G; c may lie on several blocks, but at least one, call it B_0, contains a cut-vertex $c' \neq c$ lying on F_1. Both of the faces F_i are bounded by a facial walk W_i, and each W_i contains a unique simple subcycle C_i, lying in B_0 and containing c' and c, respectively. If G has cut vertices $c_1, \ldots, c_i$ lying on F_1, we label the blocks other than B_0 incident with $c_1, \ldots, c_i$ $B_1, B_2, \ldots, B_j$, and the blocks $D_1, \ldots, D_k$ incident with c. Then we prove by induction on j that there is a PVG layout L of G with F_1 represented by the exterior face of L, with (a_c, a_d) a long-arc pair at the origin for some neighbor d of c, and with the blocks incident with c represented within $C(a_c)$.

Theorem 3.7. If a simple graph has an embedding on the projective plane with all cut-vertices on a common face, then it is a PVG.

Proof. Let G have an embedding on the projective plane $\boldsymbol{P}$ with all cut-vertices on a common face F. We prove by induction on $n = |V(G)|$ that G has a PVG layout with arcs representing cut-vertices on the exterior face and with its embedding equivalent to that of G. When $n < 5$, the graph has a BVG layout and so a PVG one by Prop. 3.1; each such graph containing a cycle also has a 2-cell embedding on the projective plane and an equivalent PVG layout.

If G has no cut-vertex, then we apply Prop. 3.3(iii) for graphs on the projective plane to get the PVG layout of G.

If G has a cut-vertex, we consider the block-cutpoint tree T of G, and, if possible, let c be a cut-vertex incident with a leaf of T with that leaf-block planar and embedded in a contractible region of P; call this block B. Deleting the vertices and edges of $B \setminus \{c\}$ leaves G' on the projective plane with face F now a face F', containing all remaining cut-vertices. By induction G' has a PVG layout L' that is equivalent to G' and with exterior face representing F'. Then there is a BVG layout of B with the bar representing c bottommost and extending the width of the layout, and by Prop. 3.1 B has a corresponding PVG layout L_B. Then a_c in L_B can be inserted as a subarc of a_c on the exterior face of L' so that L_B together with L' gives the desired layout of G.

Otherwise every leaf-block B is embedded in a noncontractible region of $\boldsymbol{P}$ and contains a noncontractible cycle in its embedding. If blocks B and B' are two such leaves, they must intersect at a cut-vertex c since every pair of noncontractible cycles on $\boldsymbol{P}$ intersects. If there are additional blocks, there are additional leaves which must also all meet at c so that T is a star $K_{1,i}$ with the non-leaf vertex of T representing c, the only cut-vertex of G, and each block is embedded in a wedge of $\boldsymbol{P}$, all wedges meeting at, say, the origin. Such a graph is planar with one cut-vertex c and so by Theorem 3.6 is a PVG.

Proof of Theorem 2.2 for simple graphs. By Theorems 3.6 and 3.7 the graphs described are PVGs. Conversely if L is a layout of a PVG G, then G has an embedding on the projective plane by Prop. 2.1 with embedding $E(L)$. If L has no visibility through the origin, then by Prop. 3.1 G is a BVG and so embeds in the plane with all cut-vertices on a common face. Otherwise, if L contains a long arc, satisfying the conditions of Cor. 3.5, then G embeds in the plane with all but one cut-vertex on a common face. Otherwise G embeds in the projective plane with all cut-vertices on a common face by Prop 3.4.

4 Results on CVGs

As the example in Fig. 1b and its extensions demonstrate, cut-vertices on many faces can be achieved using circles in layouts. We characterize CVGs in this section, as given in Theorem 2.3.

Suppose G has a layout L with circles $c_1, c_2, \ldots, c_k$ at radii $r_1 < r_2 < \cdots < r_k$ and with no circle replaceable by an arc so that the same visibilities are achieved. The circles c_i divide up the plane into annular regions and one projective planar region; note that neither the interior of c_1, denoted $\mathrm{int}(c_1)$, nor the exterior of c_k, $\mathrm{ext}(c_k)$, is empty in L since neither circle can be replaced by an arc. Then the corresponding vertices $v_1, v_2, \ldots, v_k$ of G are cut-vertices, and G is the union of graphs whose layouts lie in the annular regions plus the innermost region: $G = G_1 \cup G_2 \cup \ldots \cup G_k \cup G_{k+1}$ where G_1 is the subgraph whose layout in L lies on $c_1 \cup \mathrm{int}(c_1)$, G_{k+1} lies on $c_{k+1} \cup \mathrm{ext}(c_{k+1})$, and for $i = 2, \ldots, k$, G_i lies on the annulus given by $c_{i-1} \cup c_i \cup (\mathrm{int}(c_i) \cap \mathrm{ext}(c_{i-1}))$. Thus $G_2, \ldots, G_{k+1}$ are each planar. In addition for $i = 2, \ldots, k$ G_i is 2-connected since each block of G_i contains some vertices adjacent to v_{i-1} and some to v_i. Thus the block-cutpoint tree for G, $BC(G)$, contains a path of $2k-1$ vertices, representing consecutively $v_1, G_2, v_2, \ldots, G_k, v_k$. What sorts of graphs are possible for G_1 and for G_{k+1}, and what additional tree structure in $BC(G)$ is possible at the two ends of this path?

Consider G_1, laid out on $c_1 \cup \mathrm{int}(c_1)$, with c_1 opened up to become an arc a_1 so that this is a PVG layout of G_1. If G_1 is planar, by Prop. 3.4 and its proof, G_1 can have at most one additional cut-vertex, not on the exterior face but represented by a long arc a^* at radius 1. If there is no long arc a^* besides a_1, then v_1 may be attached to an arbitrary positive number of planar blocks. If there is a long arc $a^* \neq a_1$, then each block represented between a^* and a_1 sees these two arcs and so there is only one block lying in this annular region. Inside and attached to a^* may be any number $i_a \geq 0$ of 2-connected, planar graphs, but in any case, $BC(G)$ has attached to the path-end v_1 either $i_1 > 0$ leaves or else one additional block vertex b, representing part or all of G_1, then a vertex for a^* that is also adjacent to $i_a > 0$ vertices of degree one. (Thus the latter case corresponds to having v_3 represented by c_1 and v_1 by a^*.) If G_1 is not planar, by Prop. 3.4 and Cor. 3.5 it is 2-connected so that the path of $BC(G)$ is extended at v_1 by one additional vertex representing G_1.

The layout for the planar graph G_{k+1} lies in the infinite region, $c_k \cup \mathrm{ext}(c_k)$. In this layout of G_{k+1} the circle c_k can be opened up to a long arc with empty interior to form a PVG layout; by Prop. 3.4 G_{k+1} has all its cut-vertices on a common face, the exterior face, and so can have arbitrarily many cut-vertices with arbitrarily many connected blocks, provided all cut-vertices lie on the infinite face. Thus attached to v_k in $BC(G)$ is any tree representing a

planar graph with all cut-vertices, except possibly for v_k, on a common face. These remarks prove the necessity of Theorem 2.3.

Lemma 4.1. Let L be a layout of a PVG G with n vertices and with a long-arc pair at infinity or at the origin (or both). Then L can be laid out as a CVG with a circle on the exterior face at radius n or a circle about the origin at radius 1 (or both).

As noted in Section 3, a long arc at radius 1 or at n can be extended to a full circle, changing no visibilities.

Proof of the sufficiency of Thm. 2.3. Suppose G has BCHGI satisfying (1a) and (2) so that BCHGI is Hb_0, e_1, e_2, …, e_{2k+1}, TI where for $i = 1, …, k$, each e_{2i-1} represents a cut-vertex v_i of G, each e_{2i} represents a 2-connected planar graph, b_0 is a 2-connected projective planar graph, and T represents a plane graph with all cut-vertices on a face F. Such a graph embeds on the projective plane; in the layout each cut-vertex v_i will be represented by a circle c_i.

By Prop. 3.3(iv) the projective planar subgraph of G corresponding to b_0 has a PVG layout $\hat{L}_0$ with the arc a_1 representing v_1 in a long-arc pair at infinity with some neighbor of v_1. By Lemma 4.1 $\hat{L}_0$ can be changed to the CVG L_0 so that a_1 becomes a circle surrounding L_0. By Prop. 3.3(ii) the planar subgraph of G corresponding to e_2 can be represented as a PVG $\hat{L}_1$ with a_1, representing v_1, part of a long-arc pair at the origin and with a_2, representing v_2, part of a long-arc pair at infinity. By Lemma 4.1 $\hat{L}_1$ can be changed to the CVG L_1 so that a_1 and a_2 each become circles inside and surrounding L_1 respectively. Then L_1 is joined with L_0 by identifying the two copies of the circle a_1, placing L_1 wholly outside of L_0. This process of expansion can be repeated for e_4, …, e_{2k}. Finally by Prop. 3.3(i) T can be laid out as a PVG with v_k represented by a_k, part of a long-arc pair at the origin. Again by Lemma 4.1 a_k can be extended to a full circle inside of T's layout and can be identified with the circle representing a_k on the exterior of the layout previously constructed. In this way G is laid out.

If BCHGI satisfies (1b) and (2), it can be laid out similarly, only differing within c_1.

Since v_1 is incident with one or more planar blocks, we can lay these out in radial segments within c_1. Each planar block can be represented as a BVG with v_1 represented top-most and a neighbor bottom-most, then as a PVG via Prop. 3.1, and then inserted with v_1's arc as a subarc of c_1 within a distinct wedge of, say, 0 £ q £ p, giving the desired visibilities. Thus in all cases the graph can be laid out as a CVG.

5 Concluding Thoughts

It is clear that more complex graphs can be achieved in the polar visibility model by allowing visibility through the origin and diagonally across the boundary of a disc with antipodal points identified; call such a layout a doubly polar visibility layout and the resulting graphs doubly polar visibility graphs (DPVGs). These naturally lead to graphs that embed on the Klein bottle, the nonorientable surface of Euler characteristic 0. Analogous proofs to those given on the projective plane give the following results.

Proposition 5.1. a) A DPVG embeds on the Klein bottle.
b) If G has a layout L as a DPVG with no long arcs, then *G* contains no cut-vertex.
c) If a 2-connected graph G has an embedding on the Klein bottle, then *G* is a DPVG and has an equivalent doubly polar visibility layout.

It seems that a DPVG that is neither a BVG nor a PVG can have at most two cut-vertices, represented by a long arc about the origin and at infinity.

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★

r t t r tr t l

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t stru tur -tr r pr s ts t o positio o bi o t r p it r sp t to its tri o t o po ts. -tr s b i tro u b i ttist ssi] si t b quit i port t i t l o r p l orit s. or ti l p p rs usi -tr s l i t t t b i pl - t i li r ti usi o i tio o t l orit b op ro t rj] or o posi r p i to its tri o t o po ts. o r o orr t li r ti i pl t tio o it r tri o ti it o positio or -tr s is k o to us. r s o t i or- r t ss o t op ro t rj l orit] orr t t ult p rts. s rib t r l tio s ip b t -tr s tri o - t o po ts ppl t r sulti l orit to t o put tio o -tr s. ur i pl t tio is publi ll il bl i].

t tr ct r tr r pr t t co po t o o co ct r p t r p ct to t tr co ct co po t . tr tro c tt t [ ] t t c c ro t. [ ] t t or tr or r to r pr t t t o ll pl r o pl r co ct r p .

c t tr ol to port t t tr ct r t l o r p l or t . l r t l or t t t or or tr co ct r p o l c t to or or co ct r p tr (. . [ ] . t t t l to r pr t t t o ll pl r o pl r r p . . or r to opt p c c cr t r o r ll pl r [2 3] or or t t cl t r pl r t [ ]. c ro t tr r l or r t o o l r p l or t l t tr co ct t tr t clo r p tr pl r t t t [2]. r r tr ct o r tt t o to t t t c ro t.

t t or t c l p p r (. . [ ] t t or t to co tr ct t t tr ct r tr l r t " r t o o t l or t o [ ] or t tr co ct co po t o r p ...[ ]". o r to o r o l o corr ct l r t pl t t o p l c ll l l . o l corr ct pl t t o o tr r r o p rt o ool t [2]

★ rti ll support b - r t u 29/3- ors u ss rpu kt " i t l orit ür iskr t robl u i r u .

r tr r co ct o t r c o l or t
to co p t ort o o l r o co ct pl r r p t t
r o . o r t pl t to o ot r l r
t [ ].
r pr t l r t pl t to o t t tr ct r
tr . o t r l to p t tr tr co ct co po
t o t corr ct o t l or t pr t [ ] or co
po r p to t tr co ct co po t . lop corr ct l or t
or tr co ct t co po t o corr ct r pl c t lt p rt
[ ] ppl t to t co p t to o tr . r pl t t o (
r l or p l c ll l l [ ] (ct o .
p p r tr ct r ollo . c o tr tr co ct
co po t r cr ct o 3. l or t or co p t tr
tr co ct t co po t o cr ct o t lt p rt
o t opcro t rj l or t r o ct o r l o po t
o t t corr ct o . c r ll t t o r pl t t o .
o p t t o l r lt co c r r t r cr ct o .

t G $(V, E$ t t V t o rt c
E lt t o or r p r $(u, v$ t u, v V. $(v, v$ c ll
. $(u, v$ E occ r or t o c E t c ll
. G c ll t co t t r l loop or lt pl . E
t o $V(E$ ot t t o ll rt c c t to t l t o
E. p v w G q c o rt c l
ro v to w. p t ll t rt c r t ct. p v w
pl p t t p pl t $(w, v$.
r ct lt r p G $(V, E$ r p r v, w V o
rt c G co ct p t . co ct lt r p G
or c tr pl o t ct rt c v, w, a t r p t p v w c t t a
ot o p. t G $(V, E$ co ct lt r p a, b V. E c
to q l c cl E , . . . , E_k c t t t o c l o
co o p t ot co t rt o a, b c pt po t r t
cl . cl E_i r c ll t o G t r p ct to
a, b . t r r t l t t o p r t o cl t a, b
o G l (t r r ctl t o p r t o cl o cl co t o
l or (t r r ctl t r cl c co t o l
. G co t o p r t o p r G c ll .
T r ct r p o rl r ct r p co ct
c t t t r ctl o rt (c ll t o co
r ot r rt ctl o co . T ro v to w
ot t v w. t r (r ct p t ro v to w rt v w.
v w v t o w w o v. v w v
o w w o v. r rt c tor c t o

t l. G r ct lt r p tr T o G T r p o G T co t ll rt c G.
P r ct lt r p c t t c P t r tr rc (ot t v w or ro (ot t v w t t ollo prop rt

(r p T co t o ll tr rc p tr o P.
(v w t w v.

t G $(V, E$ co ct lt r p a, b p r t o p r o G E $, \ldots, E_k$ t p r t o cl o G t r p ct to a, b . t E $\bigcup_i^{\ell}$ E_i E $\bigcup_{i \ \ell}^{k}$ E_i c t t E 2 E 2. t o r p G $(V(E$ $, E$ e G $(V(E$ $, E$ e r c ll o G t r p ct to a, b r e $(a, b$. pl c lt r p G t o pl t r p c ll pl tt G. c pl t r p co ct . e c ll t t pl t op r t o .
ppo G pl t t pl t r p r pl t o o t l o or pl t op r t o r po l . r lt r p r c ll t o G. c o t t o t r lt pl (or c cl o l t t r (or tr co ct pl r p . pl t co po t r ot c r l q .

G $(V, E$

(E

([ ] $3\,E$ $-$

t G $(V\,, E$ G $(V\,, E$ t o pl t co po t co t t rt l e. r p G $(V$ $V, (E$ E e c ll o G G . pl c t o co po t G G r r p o G G c ll G G .
o G r o t ro t pl t co po t r t tr pl o to l t o lt pl (t tr l to l pl c cl (.

[ ] G

r co ct co po t o r p r clo l r l t to tr .
tr r or ll [ ] or pl r r p o l . r c t t or r l t o [ ] t t l o ppl to ot c r l pl r r p .
t G co ct r p . o G t r p r t o p r or p r o j c t rt c . o pl t p r u, v t r $(u, v$ or l r p C o G c t t u, v ot pl t p r o C. t s, t pl t p r o G. u, v o G t r p ct

to s,t c t t or ot r pl t p r u,v rt c u v s t r t pl t co po t.

t e $(s,t$ o G c ll t . tr o G t r p ct to e root or r tr o o r o o r t p . c o μ o oc t co ct lt r p c ll t o μ. r r c r l ollo

G co t o ctl t o p r ll l t s t t co t o l o o l to G t l.

t pl t p r s,t t l t t r pl t co po t G ,..., G_k t root o o μ o l to co t o k p r ll l e e ,..., e_k t s t.

t r t pl t p r s,t ctl t o pl t co po t o o t e t ot r o ot t G. G c t rt c c ,..., c_{k-} $(k$ 2 t t p rt t o G to t loc G ,..., G_k t or r ro s to t t root o o μ o l to t c cl e , e ,..., e_k r e e c s c_k t e_i $(c_{i-}$, c_i $(i$,..., k .

o o t o c ppl l t s , t ,..., s_k, t_k t l pl t p r o G t r p ct to s,t $(k$ or i ,..., k l t G_i t o o ll t pl t co po t o s_i, t_i t t o co t e. root o o o l to o t ro G r pl c c r p G_i t t e_i $(s_i, t_i$.

c pt or t tr l c μ c l r μ ,..., μ_k c t t μ_i t root o t tr o G_i e_i t r p ct to e_i $(i$,..., k . po t o e_i r c ll t o o μ_i. rt l o o μ_i e_i o l to o μ. r co pl t o r pr t t r r c e t t p r t o μ o t t t co t root.

c G oc t t o . c e_i l to o μ oc t t t c l μ_i o μ. t po l to root t r tr r o μ r lt tr t r p ct to t oc t t μ [ ]. o r pl t t o l tl r t t q l t t o o tr . o t o t t

t l to r p t . t l to o μ c oc t t o t or l t o r l t t ot oc t t c l o μ ll ot r l to r rt l oc t t or o . t o t o c o t t t l to r p r t q tr co ct co po t o G

• • • • • • • • • G

(G

(μ,ν

(G

(tc r r t t u, v p r t o p r t pl t co po t o u, v r t p r t o cl t r p ct to u, v . t p r ll l r r c o t t o o tr r p G , . . . , G_k r co r . t t G , . . . , G_ℓ co t or t o G_ℓ , . . . , G_k co t ctl o . c o t t r c t r c r co po t o t p c r l p r or ℓ pl t op r t o c pl tt o o G_i i ℓ tro c rt l e t l to o o μ t l to o c l μ_i o μ. c e r t l to o μ_i q t t p p rt (o t t or ollo .

l l to r p r c t r pol o o or pl tr co ct r p o t o pol o o t o o r co o rt l . t l to r p r t q tr co ct co po t o G pro p rt (. l t p rt o t t or ollo r ctl ro (.

• • • • • •• • • •• • •

t G co ct lt r p t o t l loop . ccor to or t c to co p t t tr co ct co po t o G c o or t o to l t tr o G. corr ct t lt p rt t l or t opcro t rj [ ] ppl t o l or t or co p t t tr co ct co po t . oc o t co p t t o o pl t p r c t cr pt o o t p rt [ ] ot o l co t co t l o r rror . or o r o t opcro t rj l or t pl r r to [ ] or [3].

• •• • • • • • •• • • • • • •• • • • •

p t to t l or t co ct lt r p G $(V, E$ r r c e_r. t r t t p l o lt pl r r pl c rt l o l . . cr t t o o C , . . . , C_k r lt pl r p G . r q r ort o t l • •• c o $(V + E$ t c t ort t o t . r tl ccor to t po t t lo r co l to t o t r r t t rt c q c t r , . . . , V . or loop l • •• t r t o r ll o l . r t $(V + E$.

• •• • • •• • • • • pl t o lt pl
• •• *t s s t t t s t t*
• •• • •• *t s* $e_\bullet, \ldots, e_\ell$ *t* ℓ 2 • •
• •• $e_\bullet, \ldots, e_\ell$ b s b t v w
$e_\bullet, \ldots, e_\ell$ e' (v, w)
t *t* C $e_\bullet, \ldots, e_\ell, e'$
• • •

co t p t pl t co po t C_k $,\ldots,C_m$ o G. pro c r pr t t l t t ct o . tr co ct co po t o t p t r p G r cr t p rt ll r l t co po t C $,\ldots,C_m$. lo t o o or t o pol o C_i C_j co t t rt l t C_i C_j r r . o l . 2. o co po t r r pt . or ll loop l t p o r C_i . . t o to C_i r t loop. t t l c o co t t t pr co p t or c rt l e t t o co po t to c e lo . r pr t t co po t C_i l t o c llo to pl t t t op r t o l co t t t . ccor to t tot l r o ll co po t (E o l . 2 c l o pl t t (V + E .

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l tr co ct co po t

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i m

C_i C_i s

e C_i

t sts j i t e C_j $type(C_i)$ $type(C_j)$

C_i $(C_i$ $C_j)$ e

C_j

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pr c t p o or t o to l t tr o G. ppl or t to co tr ct t root r o o . c o t o o r r pr t t o root t t o o l to co t t r r c e_r. r t co tr ct o l o cr t cro l t c tr μ ν t t o corr po rt l l to o μ l to o ν.

Suppo p l tr P or t pl co ct r p G $(V, E$ t rt c o G r r $,\ldots,$ V. t ollo t rt c t t r r . tro c t ollo ot t o

$$(v \quad \Big(v \quad w \;\; v \quad w \Big)$$

$$2(v \quad \Big(v \quad \Big(w \;\; v \quad w \qquad (v \;\; \Big)\Big)$$

t $(v$ t rt r c l tr r ro or or tr rc ollo o ro o P (or v o c rt t $2(v$ t rt r c l t (or v o c rt t .

ot t $(v$ t or r (o c clc j c c l t o rt v t $D(v$ t t o c t o $v.$ or r o t rt c or r o t t j c c l t t t ollo prop rt

(t root o P .
(2 v V w $,\ldots,w_n$ r t c l r o v P ccor to t or r $(v$ t w_i $w+$ $D(w_i$ $\ldots$ $D(w_n$ $+$.
(3 t e $(v$ r c or r ccor to $(w$ e v w or w e v w r p ct l .
t w $,\ldots,w_n$ t c l r o v t $(w_i$ u t or r $(v$. t r t i c t t $2(w_i$ $<v$ or i i $2(w_j$ v or i $<j$ $n.$ v u E t v u co $(v$ t v $w_i.$ v $w_i.$.

t o [ ] o to co p t c r o t rt c or r o t j c c l t l rt . l [ ] t t ro v w co t E t co t v $w_i.$ v $w_i.$ $(v$. c l o pt t ort ct o ϕ [ ]

$$\phi(e \quad \begin{cases} 3 \quad (w \quad e \quad v \quad w \quad 2(w \ <v \\ \quad 3w+ \quad e \quad v \quad w \\ 3 \quad (w \ +2 \quad e \quad v \quad w \quad 2(w \quad v \end{cases}$$

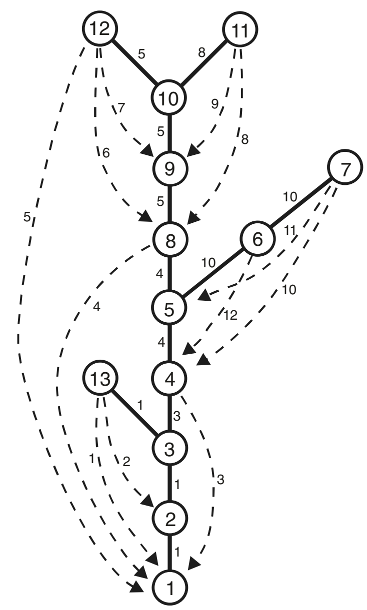
r q r or r c o t ort t ccor to t r ϕ l c t ort. or r ϕ proc r t [ ] ll ot r co ll lt pl t ot corr ctl co p t t pl t co po t o G .
ppo p r or pt r t rc o G t or r o t t j c c l t . G to t o p t co t o ro or or tr rc ollo o ro . r t p t t rt t rt p t t r t ro o t p t r c (. . c p t t t lo t po l rt o l t t l t r l rt co o t pr o l tr r p t . ro c c p t p v w c or c cl t tr p t ro w v to p (co p r [ ] .

. o p l tr t r t t t ((3 . r r ccor to t r t p t . r t p t r

2 3 3 2
2 3 2
3 3

2
2 2

o or t o u_n o u u u_n
c u_i u_i t r t $(u_i$. t q l co r p l



• •• • • • l tr it u b r rti s r t p t s.

tr P t ((3 . ollo l t r to c c co t o or p r t o p r .

• •• • • •• (3 [ ] G $(V, E$ a, b G $a < b$ a, b

r a, b s a, b b r $(r$ a $2(r$ b s r 2 r b a r b b r a x y r $x < b$ a y x y $a < y < b$ b w x $(w$ a $(a, b$ G G

2 o r t p l tr ro . . t ollo p r t o p r

t p p r (, ,(, ,(, ,(, ,(,3
t p 2 p r (, ,(, 2

r t l or t t r p G_c p l tr P_c o G_c. ot t $(v$ t r o v G_c t v w tr rc P_c t v ‹ w ro P_c t $(v$ t p r t o v P_c t $(v$ t r o c t o v P_c. c t t pl t co po t C pl t t o G_c P_c r p t . t ollo p t ct o

C new_component(e $, \dots, e_\ell$ co po t C e $, \dots, e_\ell$ cr t e $, \dots, e_\ell$ r r o ro G_c.
C C e $, \dots, e_\ell$ t e $, \dots, e_\ell$ r to C r o ro G_c.
e new_virtual_edge(v, w, C rt l e $(v, w$ cr t to co po t C G_c.
make_tree_edge(e, v w e $(v, w$ tr P_c.

or o r t cc ct o

$$(v \quad \text{r t c l o } v \quad P_c \text{ ccor to } (v .$$

$$(w \quad \begin{cases} & F(w \\ \text{o rc rt o r t t} & F(w \text{ ot r} \end{cases}$$

r $F(w$ v v ‹ w E_c t o t c or c t l ct o push pop top r

co t lr t t t r ot t to pl t co po t.

co t tr pl $(h, a, b$ (or p c l o t c r r c t t a, b pot t l t p 2 p r t o p r h t t r rt t co po t t t o l pl t o .

l or t t rt c ll t r c r proc r PathSearch or rt t root rt o P (l . 3 . r t r ro t c ll t lo to t l t pl t co po t r o .

pl t co po t
.push( )
PathSearch( )
e, …, e_ℓ b t s o
C new_component(e, …, e_ℓ)

roc r PathSearch o l . . t t or p r t o p r ppl 3 p ct p r t l l . or t p 2 l . or t p p r t o p r . or t l cr pt o o t l or t pl r r to [3]. or r to c l r r t t p t ollo t tr ct r

l orit ill ot s p r tio p irs but o l t s p r tio p irs or i i i t r p i to its split o po ts

```
• •••••• •  •• t rc (v
   ••••• e Adj(v) ••
      •• e v w ••••
         •• e st ts t ••••
                  (h, a, b) t a > t (w) ••••••
            •• t s t ••••
               •••••• .push(w + (w) − , t (w), v)
            ••••
               y h (h, a, b) t ••••••
               ••• (h, a, b) st t t
               •••••• .push( (y, w + (w) − ), t (w), b)
            •••
            •••••• .push( )
         •••
         PathSearch(w)
         •••••• .push(v w)
                  t s
                   t
         •• e st ts t ••••
                      t s •••••• t
         •••
••       • •••• (h, a, b) •••••• s a v b v (v) > h ••
            •••••• .pop()
         ••
      ••••
         ••• e v ‘ w
         •• e st ts t ••••
                  (h, a, b) t a > w ••••••
            •• t s t ••••
               •••••• .push(v, w, v)
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               y h (h, a, b) t ••••••
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               •••••• .push(y, w, b)
            •••
         •••
         •• w t(v) ••••
            C new_component(e, w v)
            e′ new_virtual_edge(w, v, C)
            make_tree_edge(e′, w v)
         ••••
            •••••• .push(e)
         •••
      •••
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• ••••••• • ••c c or t p 2 p r

• • ••• v $(((h, a, b)$ •••••• s a $v)$ ((w) 2
st $(w) > w))$ • •
•• a v $t(b)$ a ••••
•••••• .pop()
••••
e_{ab} nil
•• (w) 2 st $(w) > w$ ••••
C new_component()
t $s\,(v, w)$ (w, b) •••••• t C
e' new_virtual_edge(v, x, C)
•• •••••• .top() (v, b) •••• e_{ab} •••••• .pop()
••••
(h, a, b) •••••• .pop()
C new_component()
• • ••• (x, y) •••••• s a x h a y h • •
•• (x, y) (a, b) •••• e_{ab} •••••• .pop()
•••• C C •••••• .pop()
• •
e' new_virtual_edge(a, b, C)
• • •
•• e_{ab} nil ••••
C new_component(e_{ab}, e')
e' new_virtual_edge(v, b, C)
• • •
•••••• .push(e'); make_tree_edge(e', v b); w b
• • •
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• ••••••• • ••c c or t p p r

••• •• $t2(w)$ v $t\,(w) < v$ ($t(v)$ v s t t
t t s t t) ••••
C new_component()
• • ••• (x, y) •••••• s w $x < w+$ (w) w $y < w+$ (w)
• •
C C •••••• .pop()
• •
e' new_virtual_edge($v,$ $t\,(w), C$)
•• •••••• .top() $(v,$ $t\,(w))$ ••••
C new_component(•••••• .pop() e')
e' new_virtual_edge($v,$ $t\,(w), C$)
• • •
•• $t\,(w)$ $t(v)$ ••••
•••••• .push(e')
make_tree_edge($e',$ $t\,(w)$ v)
••••
C new_component($e',$ $t\,(w)$ v)
e' new_virtual_edge($t\,(w), v, C$)
make_tree_edge($e',$ $t\,(w)$ v)
• • •
• • •

. ut r . ut l

- p l tr P r pr t rr •••••••• ••••_•••• (t tr rc t r v •••••• (tr rc or ro .
- l $(v$ $2(v$ $(v$ r pr co p t . t ot c r to p t t .
- rr •••••••• co t t r o v G_c. t p t c t to or r o ro G_c.
- or r to co p t $(v$ p t t j c c l t c t to or r o ro G_c.
- or r to co p t $(v$ pr co p t t l t o ro v_i w t w t or r t r t . ro r o ro or to G_c t r p ct l t p t .
- pr co p t rr ••••••• c tr e t rt p t .
- t t "v j c t to ot t t tr rc" l ••• c o pl co t t t j c t tr rc .

• •••••••••• •• ••• • •••••• ••• ••••• • •••••••

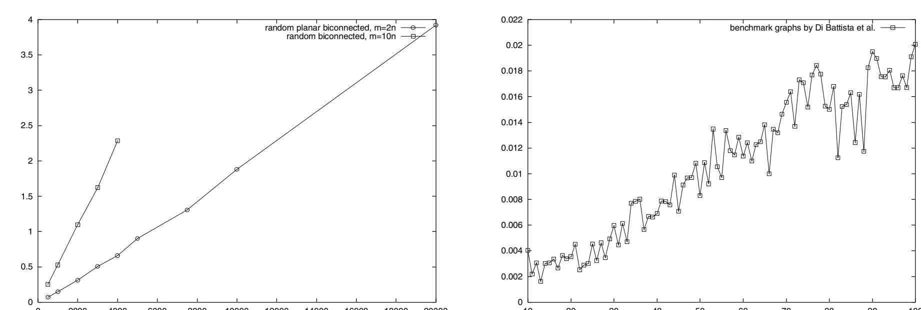
roc r [ ] o ot corr ctl pl t r p to t pl t co po t . r t port t c o r l or t

- ort ct o ϕ to o cr ct o .2 or r to t ll lt pl .
- cr t o o t l t pl t co po t (l •• .
- co t o l •• c . or l co t o co l r o tr pl ro •••••• corr po to r l t p 2 p r t o p r . c p r t o p r co l ot r co t or l proc r .
- co t o l •• c . or l co t o co l cor r ctl t p r t o p r t r t r p o .
- p t or $(v$ (c $(v$ [ ] $(v$ r ot c t.
- $(w$ (c $(w$ [ ] ot p t c ot corr ct. t c r to p t c ll G_c o . r pl c $(w$ l t o ro t w c p t G_c c .

• ••• •••••••••••••••• ••••

r pl t t o o [2] p l c ll l l [ ]. t t o r pl t t o t r t pl r o pl r co ct r p t c r r p coll ct tt t t l. [ ] r ro to rt c . pl r co ct r p t n rt c m r t n r o l c o pl t $m-n$ pl t c op r t o . r l co ct r p r t cr t r o r p G G co ct t . co p t tr r to t c ll c c t r l co t c pl l t t t .

r r t r p ct . 2. x o t r o rt c t y t r t co . l t o t r lt or r t r p t n rt c m ppl t l or t to t t t c c . r l r t c t co l ol t l t co . r t o t r lt or t c r r p ppl t l or t to c co ct co po t co t t l t t r . r p t rt c too o t . 2 co . c tl t r t l rc r [ ] pr t l r t l or t or ol t o rt o pro l opt ll (r o cro o r ll co tor l . r l or t r q r to co p t tr or o co ct co po t o pl r p t r p . t r t t t l o t c r r p ro [ ] ppl o r pl t t o o tr . lo t r t or r p t rt c . co r 3 to rt .



• •• • • • r ru i ti s (s st o ur tio tiu 2)

• •••••••••

] s s b. 2 . - l k- stitut rbrü k is i rsit¨t i i rsit¨t u öl i rsit¨t ri r. lso `http://www.mpi-sb.mpg.de/AGD/`.

2] . i ttist . ssi . -li r p l orit s it -tr s. . . t rso itor t $7t$ t t q t t s olu 3 o t t s t p s 9 6 . pri r- rl 99 .

3] . rtol i . i ttist . i i o. o puti ort o o l r i s it t i i u u b r o b s. t s t s t t t olu 272 o t t s t p s 33 3 99 .

] . rtol i . i ttist . iott . i o. pti l up r pl - rit t sti o si l -sour i r p s. t 27( ) 32 69 99 .

] . i sto k . . o . t o pl it o b i pl r r p s to i i i rt i ist sur s. *t* ( ) 93 9 99 .
6] li s l us. li r ti l orit to r o i lust r pl r r p s its p r ll li tio . . . u si . . our itors ’ *t* *t s* *t* *s* olu 3 o *t* *t s* *t* p s 239 2 . pri r- rl 99 .
7] . i ttist . r . iott . ssi . r iu. p ri t l o p risio o our r p r i l orit s. *t* 7 3 3 326 997.
] . i ttist . ssi . r t l pl rit t sti . *3 t* *t s* *t* p s 36 9 9.
9] . i ttist . ssi . -li i t o tri o t o po ts it -tr s. *t* 3 2 3 996.
] . i ttist . ssi . -li pl rit t sti . *t* 2 ( ) 9 6 997 996.
] . i i o. ip rti to i or ti uto io i rsit` i o r o t l rso l o u i tio .
2] *t* *t t* . `http://www.dia.uniroma3.it/~gdt`.
3] . ut r . ut l. li r ti i pl t tio o -tr s. - i l r port is i rsit¨t i 2 . o pp r.
] . ut r . ut l . iskir r. s rti i to pl r r p . *s t* *t* *s* *s t* *t s* ’ . r ss 2 . o pp r.
] . . op ro t . . rj . i i i r p i to tri o t o po ts. *t* 2(3) 3 973.
6] . . op ro t . . rj . i t pl rit t sti . *t* 2 9 6 97 .
7] . t. r i pl r r p s usi t o i l or ri . *t* *ss* 6( ) 32 996.
] . u r. i r r i l pl rit t sti . *t* 36 7 9 9 9.
9] . i . stru tur l r t ri tio o pl r o bi tori l r p s. *t* 3 6 72 937.
2] . l or . ¨ r. *t* *t* *t* *t* . bri i rsit r ss 999. to pp r.
2] . ut l . iskir r. pti i i o r ll o bi tori l b i s o pl r r p . . or u jols . urk r . o i r itors *s t* *t* *t* *t* *t t* olu 6 o p s 36 376. pri r rl 999.

i -i i o o is i i s i o
i t

p t t ti i i u i it i u 3 6-
p il k . u -u. . p
u t l ti i ku i it i
- p . il (i i) i i ki. i.t ku. . p
t ku i .i .t ku. . p

l it p-l li p l (t-
p t l xi l li) t l l t p i t i pl it
p l l t l . l l p i t i l t t
t l it i u p ti i t p p t
i pl t t p i t t l li t σ i i
t ll p i t . t i p p i pti l $O((n+m)$ l $(n+$
$m))$ l it t i t i ilit x li t
σ $O((n+m)$ l $(n+m))$ ti l it t t l t
i l li t σ n i t u p i t m i t
u p l l t l .

ot ti g s t o poi ts is o o t s to p r or i ogr p i or tio st s. t is r i l t t i port t o j ts i p l ls i i ti g t ir s or ot r ttri t s. o j ts to l l i p ig l p o s r s i t r st; or pl r i g i t i r t to lo tio s i ti tio l ls o ol s lt o g t r l ost s l ss i or tio or or i r s rs. r or igit l p s o l t s o s ts o poi ts r pr s ti g lo tio s o o j ts tog t r it l ls o t o j ts s o l tio to i s rt l ls to o -l l p i tl .

pro l o lo ti g l ls i p is ll t p l li g (or p l tt ri g) [9 4 6 . ppro i ti g l l (stri g o r t rs) its o i g r t gl o or l t t p-l li g pro l s t pro l o lo ti g s t o r t gl s i pl (it o st l s o t i i g m g s) i t t ( ) r t gl r pr s ti g l l o o j t s o l r to t o j t (2) r t gl s o ot o rl p ot r (3) r t gl o s ot o rl p o st l i t p. o itio ( ) ill t ti ll or l t i s it l s io .

r stri t o rs l s to t r o j t is poi t (o j t poi t) i t p. rig t ost poi t o o j t

○

is o t os s o j t poi t. or o r o l o si r is p r ll l r t gl s s l ls. [3 7 or or o pli t l li g pro l s.

t si o r t r is gi (t r or t si o l l is gi) t to i t r t r ists si l sol tio s tis i g t o itio s ( ) (2) (3) o . pro l is ll t . lso t to o si r t i i o p t t i r t r si ll or i t r is si l sol tio . or o i s s ss t t r t gl s r los t llo r t gl to to ot r r t gl s or o st l s o its o r .

isio pro l is r i g r l or g r [s o t t i t r r o r i t s o t pl t or l l it is - r i g r l to i t si ilit . it is - r i l l is it sq r st pl i t t t orr spo i g o j t poi t is t o o its o r or rs (o r-positio o l); s t t s l l is t or r. to [s o t t t pro l r i s to - r i t r r t r i t s or l l.

t ot r si i l l is it sq r pi t o o its t o l t or rs (t o-positio o l) t pro l is pol o i l ti sol l . g r l i t r r t ost t o i t s o t pl t or l l t pro l is pol o i l-ti sol l si it or l t s 2- pro l [. or o r ppro i tio lgorit s it pro l ppro i tio r tios r gi or s r l s l rsio s o t p l li g [9 4 . t s li g tor s r t q lit o t sol tio t r o l ls t t pl it o t o rl ppi g t r r lgorit s or s r l s s [9 .

l it ot r t p o p l li g ll fl r fl i l oos t s p o l l ro i t s t o r t gl s. os l ls r pl i t p t r s l s li g tor i is o o or ll l ls. pro l o i i g t si ilit o s p -fl i l l li g pro l is - r i g r l t o pl it o sol i g t pro l il p s o t r s o i t s ts. i t s t is t s t o ll r t gl s it gi r t l li g is ll so sp i l s s r i stig t t rri g i [6 .

r oti tio is s ollo s o si r r t g l r l l r pr s ti g r t r stri g o l gt . t s i t (r t r its) i it is ritt i si gl li . o r ol t l l to r s t i t . or o r i t i s (lso p s or or) l g g s st rit r t r stri g rti ll tr spos l l (i. . g its i t ig t). t is o t s t t ol i g tr spositio i pro t l li g l o t ppos t t r pr s t r t r stri g " si g t r s ori o t l i t o li s rti l. o t rst t r pi t r s ( ) ( ) ( ) o ig r ill str t s l l pl t si g si gl -s p l ls

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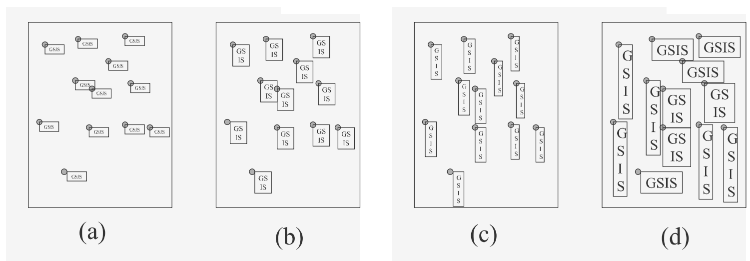
• " u t l ti i "

pi t t l t- pp r or r. pi t r ( ) s o s t i pro t o t s li g tor i s t r iff r t i s o s p s ll tog t r.



• ••• • • l t u i t ki l l . ( ) ( ) ( ) u i l ki l l ( ) u t ki l l .

r or t ollo i g *fl* pro l t r ll ris s

ppos t t poi ts poi t s s t o i t l ls o rio s r t g l r s p s pi t t pp r-l t or r. l l or poi t st s l t ro t i t s t pl t r s l . o i lt is it to t l rg st s li g tor to l l ll poi ts?

t is p p r propos l ss o s p -fl i l l li g pro l s ( ) r t i t s t o r t gl s gi to o j t poi t st . i-tio o l t-p rt or r s t ill gi i t t s tio ; t pi l pl is s t o r t gl s pi t t ir l t- pp r or rs. s t - or r-positio s p -fl i l l li g pro l is sp i l s o .

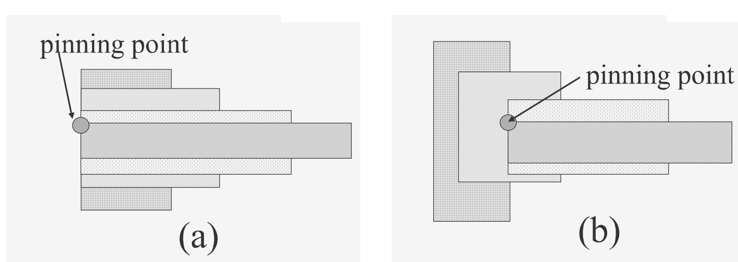
s o t t t isio pro l or sol i (($+$ $m)\log($ $+m))$ ti si pl pl -s p lgorit . s o s q t opti i tio pro l or sol i (($+m)\log($ $+m)\log\Gamma)$ ti i t oor i t l s o poi ts r r pr s t $\log\Gamma$- it i t g rs. lso gi (($+m)\log$ ($+m))$ ti lgorit or t opti i tio pro l . o sig t is lgorit s t p r tri s r p r ig i o l ; s p r ll l sort poi t lo tio q r to sig "g i lgorit r q ir or t p r tri s r .

r t o s s s ro ti i risti lgorit s or pr ti l l li g s st . o pr li i r p ri t o t ilit o to l rg t l li g si o p r to si gl -s p o ls.

• • ••••• •• • ••• •

s t o r t gl s i t pl is it r sp t to i l sio i p ir o r t gl s i s tis s it r or .
or r t gl r pr s ti g l l poi t ( ) i ll t o i t los r o . o si r s t o r t gl s it pi i g poi ts s s t o i t l ls. ppos t t pl t r t gl s i so t t t ir pi i g poi ts r tr sl t to t origi . t ¯ t los l pl . ¯ (or its tr sl t /s l op i is tr sl t /s l) is ll t o .
¯ ¯ is s t o r t gl s. ¯ is tot ll or r s t s t s tis s t ll t s t .

s t t l t-p rt or r s t is i t pi i g poi t o l l is o t l t g ; i ot r or ¯ is rti l s g t. t r is s is o - g r t . ig r 2 ( ) is pl o g r t l t-p rt or r s t ig r 2 ( ) is pl o o - g r t l t-p rt or r s t. or si pli it i l o si r g r t l t-p rt or r s t to s ri lgorit s proo s si it is ro ti to g r li t or o - g r t s .



• •• • • • ( ) t l t-p t t i ti u t l ( ) t l t-p t t.

r gi s t . . . • o o j t poi ts o t pl . t • (• •) or 2 . . . . lso o si r s t o pol go l (ot ss ril r tili r) o st l s i t pl i o l l is p r itt to o rl p. p r it l ls to to o st l s. ss t t t o st l s o ot i t rs t ot r. t m t r o g s o pol go s i . l t-p rt or r s t • o r t g l r l ls is gi to o j t poi t • . • • • iff r t ro ot r i .
t positi r l l ll . oos s it l l l • ro • or • s l • t tor pl it i t pl

so t t its pi i g poi t is lo t t .. pl t . . . .
is ll i ( ) o t o l ls o rl p ot r (2) o l l o rl ps
o st l . r i is to t l rg st s li g tor or i si l
pl t ists to o p t t pl t.
ss t t t r is o p ir o l ls i . s t t
si ot s t l rg r l l (ll) i o r sol tio .
l l s t it s p irs pr pro ss t l l s ts r o i g
ll r t l ls.
or r o t s t . s ollo s or t o l ls
i . i o l i ‾ ‾. s t t is t
i .
si g t or r t r ll gi l i ogr p i l or r o g t
s t o si l sol tio s (or) s ollo s sort t o j t poi ts
. . . . i o i r si g or r it r sp t to t - oor i t s r -
rr g t ri g; is t rig t ost poi t . is t l t ost
poi t. t (... .) (... .) t o iff r t si l
pl ts r . . . . r l ls or . .
i t r is i s t t . . . . or r
. i i si l pl t it r sp t to t is or r is ll t
sol tio .
t ∑ . is t r o ll i t r t gl s. ogr -
p i or tio st t r i lit o l t-p rt or r s t or poi t is
s ll o o st t. r or ss ( ) i t is p p r
lt o g it is ot i lt to g r li o r rg t to t s s r is
l rg r t . t isio pro l sol it o t i r -
si g t ti o pl it i t l l s t o poi t is i it s t
pro i t t i t (i pr is (log($+m$)) orti ti)
t o to q r t l t-s ll st l l t t o s ot i t rs t t "ro ti r
i tio 3.2. t pi l pl is t - or r-positio l sti l -
li g pro l r l l s t o sists o ll r t gl s it s r
l l is pi t its pp r-l t or r. ot r pl is t
l li g pro l propos r l t l.[9 .

or pr s ti g lgorit s or r r t t t l t-p rt or r
prop rt is r i l or sig i g pol o i l ti lgorit . it is -
r to o p t si l pl t os s li g tor is l rg r t − +
ti s t opti l s li g tor or positi o st t . r ss r s lt
sil o t i o i i g t r tio o t pl r 3 pro l to
l li g pro l it t r i t l ls [t proo is o itt
i t is rsio . t ot r o ot o str t "lt r ti g l
g g t r pr s ti g gr p - g i t r tio i o l l t-p rt or r s t
is llo to poi t.

• • •••••• • • • • ••• •

t is s tio pr s t $((+ m)\log(+ m))$ ti lgorit to sol t isio pro l or s li g . it o t loss o g r lit ss i t is s tio . rst sort t o j t poi ts i o i r si g or r it r sp t to t - oor i t l s r - rr g t ri g s ot or . st rt it t ollo i g o s r tio

(. . . •)
• •
. . . •_ •
(. . . •_ • . . . •)

ppos or o tr i tio t t t pl t is i si l . t r st i s t t t l l • ssig to • i t origi l pl t i t rs ts . i • is lo t to t l t o • • st i t rs t t l t-p rt o . o r • st lso i t rs t t l t-p rt o • s is l t-s ll r t •. is o tr i ts t si ilit o t origi l pl t .

ro t is o s r tio sig si pl i r t l lgorit to i t si ilit . t is l r t t is orr t it o tp ts t l t- i i sol tio i t i p t i st is si l

(* i t si ilit o i st or gi)
.
2.
3. r l l i • o rl ps o st l or l l pl so r
4. “ pro l is i si l
. ssig • t l t-s ll st l l • t t o rl ps
it r o st l or l l pl so r
6.
7.
. (. . . •) s si l sol tio

gi i pl t tio o si g st r pl s p t o s o t t it t s $((+ m)\log(+ m))$ ti . r r t t t pl s p t o is i l s i t r t gl pl t l li g pro l s; or pl r l [9 .

irst ss or si pli it t t t r is o o st l gi lgorit or t s ; ill ri fl pl i l t r o to o i it or t s it o st l s. or l l its is t ori o t l ist t

its pi i g poi t its rig t g . s pr pro ssi g sort t r t gl s i . i s i g or r o rig t-p rt i t or 2 . . . . t t s (log) (log) ti . or s t . positi r l r l t .( ) t l t-s ll st l l i . os rig t-p rt i t is t ost .

t s t o g o tri o j ts (i o r s pl l ls) i t pl l t rti l li t s t t poi t i o j t i is ro i is o t rig t o t ori o t l l-li ti g ro to t l t o s ot i t rs t o j ts o til it ts . io o ll l t- isi l poi ts o o j ts i is ll t o t .

ot [t l li g o poi ts . . . . o t i t lgo-rit . ll t t . (. .). or . . t ro ti r o [t t rti l li is io o ll l t- isi l s g ts o l t g s o l ls (r t gl s) i [. ro ti r s ( ) s g ts its ort ogo l proj tio o to t - is i s p rtitio o t is i to ( ) i t r ls (i pr is t ost 2 + i t r ls). sort list ([ ) o t s i t r ls it r sp t to t - oor i t l s o t poi ts is ll t

. o i t r l i t proj t ro ti r ssig t - oor i t l . o t s g t i t ro ti r os proj t i g is . l . is s t to i t r is o l l i [os proj tio o t i s . i pl t t list ([ ) o i t r ls si g s it l i i r-s r t str t r [. s t i t r l o t i i g . i ti (log) s t list o i t r ls i (log) ti p r i t r l.

r pl s p lgorit o s t s p li to t l t ro to − . il . . i t i t proj t ro ti r ([ ) tog t r it t l s . or ll i t r ls . t s p li o s to . i s rt l l o . to [p t ([ ) to ([+). o it t ils i t is rsio s o sp li it tio .

(log)

isio pro l is t l st s i lt s t l t iq ss pro-l [3 t (log) ti o pl it is opti l o t lg r i isio tr o l.

t r r o st l s t r or o t lgorit is t s s o s ri o . o r t ro ti r o t i s “p rts o o st l s s ll s l t g s o pl l ls. o s p rts s g t o g o o st l or o t o po t o t i t rs tio o o st l s t s p li . p rt is s g t o g t orr spo i g i t r l i t proj t ro ti r s o l o t i t q tio o t g .

jor i lt is i li g o o st l s i t rs ti g t s p li . or o t o po t o t i t rs tio o o st l s it t s p li i t r l i li rl g s i t p r t r i t proj t ro ti r. i t r l or its j t i t r l s ri li i t . r or to o si r t p o ts r i t r l i t proj t ro ti r is li i t . o r t r o s ts is ($+m$).

i t i priorit q to q r t rli st li i tio ti o i t r ls i

t proj t ro ti r. is priorit q p t i $(\log(m+ \;))$ ti p r o i s rtio l tio t- o op r tio s o i t r ls. logo sl to t l sis i t pr io s s s tio pro t t $(\;+m)$ i t r ls i t proj t ro ti r r i s rt l t or t o i t ol pl s p pro r . i t i t priorit q i $((\;+m)\log(\;+m))$ ti i tot l.

• • • •• •• • •• • • • • •• •

o si r t pro l o i g t i si l l o t s li g tor . si pl i r s r lgorit o or s; t lgorit or t isio pro l i t r s o l tr l rg r or s ll r o t t rr t s li g tor or t t s r . oor i t l s o ll poi ts r i t gr l Γ is t i o t ir sol t l t it s s to r t lgorit gi i tio 3. $(\log\Gamma)$ r o ti s (ot t t Γ). s

$$((\;+m)\log(\;+m)\log\Gamma)$$

i r s r lgorit o is i t or pr ti l i p ts or i $\log\Gamma$ $(\log(\;+m))$ ol s. o r i t lgorit it ti o pl - it i p t o Γ is sir l ro t t or ti l poi t o i . sig s $((\;+m)\log(\;+m))$ ti lgorit or t opti i tio pro l .
possi l t o is to o si r t fl o i t l ls rst list p ll possi l riti l l s o t ppl t i r s r to t list. or s r l l li g pro l s [list o si ( ) o t riti l l s o t o t o is i t. is is s t si o t o fli t gr p t … is r to ( ) or t os pro l s i " l rl s l ss l ls r o itt . ort t l t prop rt o s ot ol or t o ot o o to o t i s list o si ( ). s ppl ot r ppro .
ggi o s p r tri s r [is o s t o to tr s or pr isio - p t i r s r lgorit i to pr isio -i p t lgorit . s- p i ll t t o is q it s l i o p t tio l g o tr [2.
gi ri i tro tio to t p r tri s r p r ig (s [2 or t ils). ppos t t is - l tio o p r t r t r is l … s t t ( ) i … ( ) i … . r i is to o p t t l … . r tri s r ss s t t t ollo i g t o lgorit s or o p ti g ( ) or gi l r il l lgorit is ll i is t st st il l

lgorit to o p t ( ). ss t t t s (.) ti . ot r lgo-
rit is ll . si l t t ior o or ...
it o t o i g t l ... i oop r tio it t isio lgorit
t l ... i t o rs o t si l tio . t is t g o s to
s g i lgorit t t s p r ll l str t r lt o g o ot s
p r ll l i i o r o p t tio . t s (.) p r ll l ti it
pro ssors t si l t or ... it o t i p tti g t l ...
i (. log + . . log) s q ti l ti . ol s l r tio t o [4
o t i pro t ti o pl it to (. log + . .).
t s o si r o r pro l . o oto tio s
ollo s ( ) i o l i t r is si l pl t or t s li g tor
. s t p r tri s r p r ig r g r i g s t p r t r.
s or t isio lgorit . ort t l or o r pro l
g i lgorit it . $(\log(+ m))$ $(+ m)$ s s to
i lt to sig . o o r o t i lt opt “ t rog o s
rsio o p r tri s r . t rog o s p r tri s r p r ig
s s “ r g i lgorit t t ot o p t ( ) its l i is
gi s i p t. st o p t s ot r tio ( ) r t r g
o ( ) is ot t is l rg r t gor . r q ir o itio is
t t ( ) ( ) l s i pli s ( ) ( ) or . t iti l
gi s r t o . p rti l r ill s g i lgorit o sisti g
o p r ll l sort poi t lo tio q r lgorit s.
i o t t rog o s p r tri s r s i pli itl gi i -
gi o s p p r [i i sol pro l o t p r tri i i
sp i g tr o gr p si g p r ll l sorti g lgorit s its g i l-
gorit . ol [4 lt it t t rog o s p r tri s r i i t
g i lgorit is p r ll l sort si g sorti g t or . o r to t t-
or s o l g t is is t rst ti t t t rog o s p r tri s r
lgorit si g g i lgorit i ol i g o p t tio l g o tri pro -
r is propos .

s pr pro ssi g o o r p r tri s r lgorit pr p r poi t lo -
tio t str t r ro t s t o pol go l o st l s s ollo s rst
o str t tri g l tio ( ) o t pl i to (m) tri gl s so t t
tri gl is it r o t i i o st l or o pl t l o tsi o st l s. ll
rti s g s tri gl s i ( ) r ll o ( ). pr p r
poi t lo tio t str t r so t t t o ( ) o t i i g
q r poi t i $(\log m)$ ti . tri g l tio t poi t lo tio t
str t r o str t i $(m \log m)$ ti (.g. [3) o ot
to o str t it i p r ll l si it is i p t o t l o t s li g tor.
t : . t s t o ll i t l ls l t ( ) t s t o
or r poi ts o r t gl s i t r s l pl so t t t pi i g
poi ts o to t ir orr spo i g o j t poi ts i . t ( ) t s t o
ll rti s o pol go l o st l s i .

r g i lgorit rst o p t s t sorti g lists (( ) ( )) (( ) ( )) o t poi t s t ( ) ( ) it r sp t to - - oor i t l s t lo t s ll poi ts o ( ) i ( ) i p r ll l.

p ir o p r t r l s r ll to ot r i ( ) (( ) ( )) (( ) ( )) (2) (( ) ( )) (( ) ( )) (3) poi t i ( ) is o t i i t s o ( ) s t orr spo i g poi t i ( ) is.

… …

si l ti g o r g i lgorit o p t …. orti g o ($+ m$) l ts o ppl i g sorti g t or [2 i (log($+m$)) p r ll l ti si g ($+m$) pro ssors. poi t lo tio q r o i p r ll l or o ( ) poi ts i (log m) ti . s t g i lgorit r s i (log($+m$)) ti si g ($+m$) pro ssors or o r ppl ol s l r tio t o [4 . o r p r tri s r lgorit r s i (($+m$) log ($+m$)) s q ti l ti . s o t i t ollo i g t or

(($+m$) log ($+m$))

m

• • •• •••••• •• • •••• • • • •

pr ti l s st p l li g pro l is o t gi i or t t is t or ti ll - r . r or risti s t o s or ri t o s r o t ff ti i pr ti [4 6 . s s po r l po to sig risti s o i it ot r t o s. ppos si l l li g it s li g tor gi so t o t to i pro t tor gi g t s p o l ls. t . t l l or . i t l li g. pl o t si gl l l . to . ssig ppropri t l t-p rt or r s t . s t t . .. s i st o . s li g tor i t sol tio o t is i st is l rg r t or q l to is o t l rg r t . is o si r s "lo l i pro t ro ti i is i port t tool i t - risti s.

ppos t t r gi i st or i t s t o i t l ls or . is io o t o l t-p rt or r s ts . .. ll t is o l si it r g r s o i tio o t t o-positio o l [9 . s ot or t pro l is - r to ppro i t t opti l s li g tor it i r tio − + ; t r or

risti . sol t t o-positio si g o i tio o sol tio s o 2 r 2 is lgorit or sol i g t t o-positio l li g pro l r (t ost) t o i t r t gl s or o j t poi t. or g r [g i t i pl t tio o 2 .

or l to t r i ro i o . or . t l l or . 2 . . . s o l s l t r t pro l to . o r risti s 2 to s stit t or s or l . lso s to o str t i st o 2 . o it t ils i t is rsio . s lt r t l ppl 2 til t i r s o t s li g tor stops. si il rl o i t o -sli r (rti l sli r) o l [9 .

• • •• ••• •• • • • • • • • • •• • • • • • • • • • ••

o pr li i r p ri t to s t ilit o to l rg t s li g tor. o p r o r iff r t l li g o ls ( ) -positio o l (2) (3) t o-positio o l (4) t o-positio .

pr is or o j t poi t ssig t ollo i g i t l ls or t r sp ti l li g o ls ( ) l t- pp r pi r t gl it ig t 3 r t 4 . (2) s t o si i s o l t- pp r pi r t gl s o r 2 os ig t- r t r tios r 2 3 4 3 3 4 3 2 (t orr spo to tori tio s o 2 to 2 6 2 4 3). (3) p ir o r t gl s it ig t 3 r t 4 o o i is l t- pp r pi t ot r is l t-lo r pi . (4) s t o r t gl s o sisti g o t os i (2) t ir r fl t opi s pi t t l t-lo r or r.

r o l g r t i t gr l o j t poi ts i sq r r gio o si or o 2 4 6 . i ot pl o st l s. l s o s t r g s li g tor o r i st s or o ( ) (2) (3) (4) or o . ot t t t t l o s ot i i t t q lit o l li g o tp ts lt o g (2) s l ls it l rg r r t (3) it o s ot s t t it is tt r t (3) si l li g it rio s s p s is o t l ss ti l t l li g it si gl s p . i pr ti l p l li g i st o l portio o t poi t s t s o l gi l ls it rio s s p s.

• • •• • •• • •• • • •• • • • •

r rr t i pl t tio o i t p ri t is r t r i ; r pr p ri g or p ri t si g l rg r pr ti l i st s p r or o lgorit s ill r port t r .

pr ti l ppli tio s o t t to lgorit to pl s l ls s possi l or gi i st o i is i si l or

---

• t i l i t t t i t °.

• • • •• • • li t t l li l

	x -p iti		t -p iti	t -p iti
2		6		
	23	6 3		3
6			3	
		333	2	6

s li g tor . sig o i t lgorit s or risti s or t is pro l is i port t t r pro l .

• •• • • • • • • •

. . l . l . u i l pl t xi u i - p t t i t l u •• ( ) 2 2 .
2. . t i . l . i ti i c l (n) p ll l t p • (3) pp. .
3. . u . i . i . ku xi i l ti x t i p l l t i i *6* u ( ) 22 23 .
. . l l i ti t k t t i t ti l it •• ( ) 2 2 .
. . . p ki p l it ppli ti t l tt i p u ( ) 2 2 .
6. . tu i . u i l ti l l u t p i t p ( ) 3 6 3
. . k uli . lli u i pp t l li p i l tu u ( ) 3 3 6.
. . t tu i t t i ti l pp xi ti t l i t i u u i it (u).
. . l . t i k . l i t t l li it li i l l u •• ( ) 2 .
. . i ppl i p ll l put ti l it i t i i l l- it •• (3) 2 6 .
. . l u u p p i l .
2. . l t i ti 3 u 6 3 6 (. . . l k) ( ) .
3. . . p t u u p i l .
. . . l p ti l p l li u i ti l it u • ( ) 3 .
. . . l i t i l k p l li ( ) 3 6 33 .
6. . t . . u u i ti p i t- tu t p i l l l pl t (2) (l ttp // .l .i p . /~l / i /i x. t l)

t t t

t t t

lr k r or l orot r

ni it o on t n p t nt o o put n n o tion i n
{Ulrik.Brandes, Sabine.Cornelsen, Dorothea.Wagner}@uni-konstanz.de

o o to utili t tu p nt tion o ll i- ni u ut o p to i u li t ini u ut o pl n p in pl n in . n t pp o t tu i t n o into i i l lu t in o t p t t ont in o pl t in o tion on ll t ini u ut . p nt n l o it o -pl n o t o o- n l in o i i ll lu t pl n p it t n ul l p lu t oun i n t ini u nu o n . i p- p o i t n t n to in in i t t o t u t o ini u ut p t i pl lo u .

t t

o t t u t l tru tur l prop rt o r p . t t l. o r t t t t o ll u ut o o t r p G t po t t tr l k tru tur . t r pr t tu . . o t r p r o t t o t o l . lt ou t u r o u ut r p $(n$ t o t tu l r t u r n o rt o G. ro t tu r pr t t o t p rt t o o t rt t l tr t ut t o t l o t o or t o out t G. t to u l r p G to t r t t tu r pr t t o o t u ut o r .

pl lo ur t pl to t o o t o po t . u ut t t o rt o r p to t o o t u t . u t tur l to u l u ut r o r p pl lo ur p r t t t o u t .

t l. tro u t o l o r r ll lu t r r p . r o r r ll lu t r r p t o rt o r p r pr t r o t t ou pl lo ur . t o u t o t rt t t t r pr t ult ou l t to tr tru tur . t r o ut t t t r pr t t o p r o ro ut r r ll lu t r r p . r p o ro u ut r or pl l pl r r p or l r p .

○

t r r ro ut t tru tur o t t o u ut pl t t t r r pr t t l t pl tl r o t p r o ro ut . o o r t t t o l o r r ll lu t r r p t to tu lu t r r p u t t t o l o u l r u ut pl lo ur . t o l o t t t t or t o ro u ut t our or r ut r l o u .

o tr ut o o t p p r ollo . t. 2 3 pro o k rou o t tu r pr t t o o r r ll lu t r r p r p t l . t. o o to o tru t r r ll lu t r r p ro tu r pr t t o pro t r t prop rt o ro ut pl r r p t t l u to r pr t ll u ut lo ur . ll our t o t o or r pl r r p t t r lu t r or to t r u ut r pr t t. .

t t t t

t G u r t o t r p . t $E(G$ ot t t o o G t $V(G$ t t o rt o G. r p G to t r t po t t u t o ω $E(G$ t . or t o u t S T o $V(G$ l t $E(S,T$ v,w ; v S w T t t o t S T l t $\omega(S,T$ $\sum_{e \; E \; S,T} \omega(e$ t u o t o t t t t o u t .

ut u or r p r $S,\overline{S}$ r $\subsetneq S \subsetneq V(G$ $\overline{S}$ $V(G$ S. t S *u* *t* ut $S,\overline{S}$. *t* o *t* ut $\omega(S,\overline{S}$. t λ $_{\subsetneq S \subsetneq V \; G}$ $\omega(S,\overline{S}$ ot t u o ll t t ut $S,\overline{S}$ o G t $\omega(S,\overline{S}$ λ ll *u* *ut.* t $(G$ ot t t o u ut o G. $G(S$ ot t u r p o G u t S.

l c v $,\ldots,v_k$ qu o k 3 t t rt u t t $E(c$ v $,v$ $,\ldots,$ v_{k-} $,v_k$, v_k,v $E(G$. or u t E $E(G$ ot $G-E$ t r p $(V(G$ $,E(G$ E .

r pr t t o *t* *ut* G $($ $,\varphi$ *u* *t* *t* *t* φ $V(G$ $-$ $V($ *u* *t* *t* φ^- $($ $($ φ^- $(S$ $,\varphi^-$ $(\overline{S}$; $S,\overline{S}$ $($ ν $V($ pt φ^- $(\nu$

2 *ut* $S,\overline{S}$ $T,\overline{T}$ ro *t* or r t S T S $\overline{T}$ $\overline{S}$ T $\overline{S}$ $\overline{T}$ *t* *ut* *u* *t* or r ut *t* *ut* *u* $S \Delta T$ S T T S *t* o l ut

ut *ut* o l o ut t ro ut . $(G$ o t o ro ut t t o u ut o G r pr t tr . t t l. o t t t t o u ut o r tr r t o t r p r pr t tu r l orr po to t o ro ut . or pr l

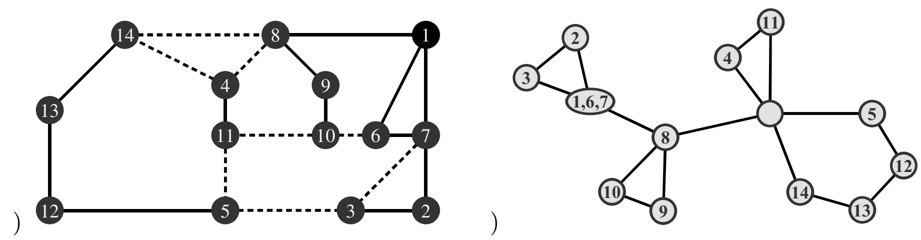
o to t ini u ut o l n p

u tu t
t t t t t t tr
t t t l

t ollo u t t t tu u . . t t ll l
t t t t r tr t t t o
l .

t $(G$ u ut t t
G t t ($,\varphi$ u t t u tu t $(n$

ur o pl o t r p t tu . t uto



• •• • • •) i t onn t p n) t tu p nt tion o it ini u
ut . n) oli i t 2 n i t . n) φ i
p nt t l l o t no .

2 r t r ll t o ut t t r pr t tu .

2 t ut t tu
t ut

t u ut
t ut t

t t tu t t t $O(n$

t ollo ot t tu r pr t t o o ll u ut
o G. ot t t t r t o t t t o u ut o
t t o tr p r o l lo to t l . u
l o t t ut G r pr t tr or p r o l
o . port t prop rt o tu r pr t t o t t ll u
l t r t t " G o ot ro l o ". or pr l

c ν $, \ldots, \nu_k$ t $_i, i$ $, \ldots, k$ t t t
t t t $-E(c$ t t t ν_i V_i φ^- ($_i$

$\omega(V_i, V_j$ $i - j$ o k.

l r o t t t tu o ll u ut o t r p o tru t $(mn \log \frac{n^{\cdot}}{m}$ t . or u t r p t o put $(\lambda n$ t 2 . t l r t ort t p t l ort t o r t l. or flo o put t o t tu o t pl r r p o t $(n$ t t t o tru t o r .

3 t

t l. tro u t r r ll lu t r r p o l r t r r p t t pl r r t r p t to t lu t r . t t o u r t o r ult o 3 t t ll u l t r.

$u\ t$ $(G, T$ o t o r p G $(V, E$ root tr T u t tt t o l o T tl V. rt o T r ll . o ν o T r pr t t $u\ t$ $V(\nu$ o l t u tr o T root t ν. T ll t u t o $(G, T$. e o G to t to lu t r $V(\nu$ e $V(\nu$.

r r ll lu t r r p $(G, T$ t lu t r u o t u r p o G.

o r r ll lu t r r p $(G, T$ lu t r o t u rl r p G o t lu o tr T t pl . rt v o G r pr t po t $(v$ e v, w pl ur $(e$ t $(v$ $(w$. o l o ν o T r pl lo r o $(\nu$ ou pl lo ur ∂ $(\nu$ u t t

$(\mu$ $(\nu$ or ll t μ o ν.
$(\mu$ $(\nu$ ϕ μ t r t or tor o ν.
$(e$ $(\nu$ or ll e o G t e $V(\nu$.
∂ $(e$ ∂ $(\nu$ l po t e $V(\nu$.

ou l p k T r t lu o r pr t t o o G o l ro lu t r ou r r .

r o e lu t r ν $u\ t$ e $V(\nu$ ϕ ut $(e$ $(\nu$ ϕ. r o r r ll lu t r r p t r r o ro o lu t r ro . r p t pl r r .

t

$u\ t$ C $(G, T$ t t G u t t ν T t $V - V(\nu$ t ut t $G(\nu$

t t u $u\ t$ o r r ll lu t r r p $(G, T$ ur $(e$ qu o or o t l rt l t or r e o G $(\nu$ p r ll l r t l or r o l o ν o T.

t $u\ t$ t n t t t t $(n$ t t t $t\ u\ t$ $(n$ t

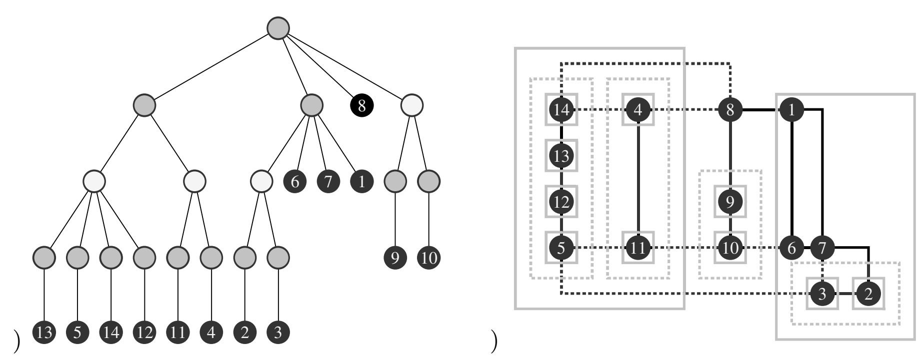
t t t t t
t

ot t tu r pr t t o o t u ut o r p t lu o tr o r r ll lu t r r p r pr t tru tur l or t o o r p . o o to tr or t tu r pr t t o to lu o tr u t t ll u ut r o r o t orr po r r ll lu t r r p .

t (, φ t tu tru tur o t u ut o t o t pl r r p G t n rt .

. or r l c ν , . . . , ν_k l t ll c (pt o ν_c ν_i, ν_c , i , . . . , k.
2. pl r pt o o r 2 t t l .
3. or r rt v o G o ν_v $\varphi(v$, ν_v .
. u t l root.

ll t t u o tru t root tr (G . ot t t (G, (G o r r ll lu t r r p . u r o o (G (n or V((n . t p o or r l t p 3 n o . u V((G r (n . ur 2 o t lu o tr (G o t r p G ro . . r r



) it no in t in lu ion t $\mathcal{T}(G)$ o t p G in i . p nt t no t t o l in $\mathcal{G}$.) o pon in lu t oun i n t n l in t -pl n in o $(G, \mathcal{T}(G))$.

r l opt o or oo root. o t root u t t $V(\nu$ $\overline{V(\nu}$ or r r o ν o (G . t t t t t o t l u ut $S, \overline{S}$ . . t ut u t t $S - \overline{S}$ l o t top l l. ot r po l t to t k t t r o t tr . . to t t. ot t root o put l r t . o u l t u ut G t to o tru t pl r r o t r r ll lu t r r p (G, (G . t l u r t t t r tr r o G lu t r ou r .

2 t t G
t t t $u\ t$ $(G,$ $(G$

o t r ll t ollo t out u ut .

t $S, \overline{S}$ u ut t o t r p .
ot $G(S$ $G(\overline{S}$ r o t .

t $S, \overline{S}$ u ut t o t pl r r p
G. o G t u l o t r p u $E(S, \overline{S}$ l .

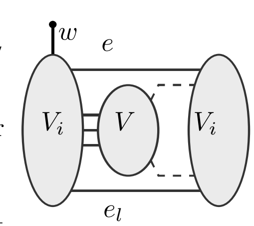
t o t u r t t t or r o t t o t
pl r r p G t r r ll lu t r r p $(G,$ $(G$ ul ll t pr o
t o o or 3 t u pl r r .

t p r pl l o t tu t r. u t or t o
out t l or r o t l o ot pr r $(G$.
o r t or r r o tru t ro pl r r o $(G,$ $(G$
ollo .

t c ν $, \ldots, \nu_k$ l l t $_i$ V_i .
o r t u l o t r p u $E(V_i, \overline{V_i}$. $V_i, \overline{V_i}$ u
ut t u l l ot d_i. ot t t t l d_i orr po to
lu t r ou r pl r r o $(G,$ $(G$. or e G l t
$\omega(e$ $\omega(e$. t o qu o

$$\sum_{e^* \quad E \ d^*_\cdot \quad E \ d^*_\cdot} \omega(e \qquad i - j \qquad o \ k$$

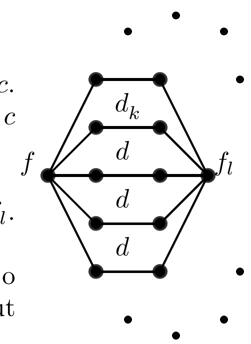
t $E(d_i$ $E(d_i$ ut d_i d_i



uppo ot. t e $, \ldots, e_l$ t qu o
p t d_i u t t e, e_l $E(d_i$ e $, \ldots, e_{l-}$ /
$E(d_i$. t e $E(d_i$ $(E(d_i$ e $, \ldots, e_{l-}$
ot r d_i l t e v, w t w / V_i. or
j $i, i+$ l t p_j p t ro e V_j to e_l V_j
t r p u V_j. t c t l G t t
u e p t p_i e_l p t p_i . t out
lo o r l t u t t e $, \ldots, e_{l-}$ r l c. t
V $V(G$ t t o rt t t r t to e $, \ldots, e_{l-}$ t t r ot
V_i. w V_{i-} V V_{i-} . u V_{i-} ot o t
o tr t rk .

$$\left| \bigcap_{i}^{k} V(d_i \right| \quad 2$$

t d f $, \ldots, f_l, f_l$ $, \ldots, f_r$ d f $, \ldots, f_l, f_l$ $, \ldots, f_s$ u
t t f_l f_l f_r f_s. ro 3 o lu t t $E(d$ $E(d$
$E(d$ $E(d$ t t $_i$ $V(d_i$ f $, f_l$. t ollo u
t l t t $_i^k$ $V(d_i$ f $, f_l$.

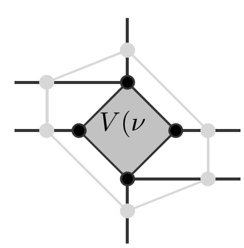


u oo o f r ll t l
G t t orr po to t ut o l c t r
t. l or r f pl t l or r o c.
or o r r ut t t r pr t t o o c
r o tru t ro pl r r o $(G,$ $(G$
t ollo . r lu t r ou r orr
po to l d_i to t o p t t f f_l.
r ult k o t p t . l o t o
t o o t p t t u l o ut r pr t t o
o c r ut t t r pr t t o ut
o c o t or .

t

u

t o t. 3 t l. 3 tro u t o or r r r
ll lu t r pl r r p ort o o ll t r t ul rl p lu t r
ou r . o o t t t u r t r p r t llo
to ro t lu t r ou r o l t t top or otto o t r t l .
t tro u u r to t r . tro u
r t o r o t pl r r p $(G, T$.
rt to G u t t t l o tru t r p G r pl r
lu t r ou r orr po to l G . o pr r
l or t ppl to r r p G t u to o t pl r r
o $(G, T$. G u r u t o l o
3 t o t o l o tr t o t flo t l to pl r
r t t u u ro .
t $(G, T$ r r ll lu t r pl r r p t
t pl t t ul ll t o t o o or 3. ll t t or T $(G$
2 r o pl r r p u t l .
or r lu t r l o rt to G t
ollo



ro ro t l to t root o T or r
o l o ν o T l t e $, \ldots, e_k$ t t to
lu t r $V(\nu$ t r l or r rou $V(\nu$. t e_k
e e_i v_i, w_i $, i$ $, \ldots, k +$. or i $, \ldots, k$
t e_i . . rt $v_{e.}$ to $V(G$ r pl
e_i $v_i, v_{e.}$ $w_i, v_{e.}$. ll
$v_{e.}, v_{e..}$. k r ll u
o $V(\nu$. or l ll t u o
$V(\nu$ t t o l t lu t r ou r o $V(\nu$.

. il i pl ntin t t n ion o i ' o l to i i ll lu t p l n t t it in p n ntl i in] n i no p t o t li].

. n . onl n n . n

p l o ur t r r o l o or t o t to lu t r. t t t o or o t o l rt r rt t l to o loop ult pl . r ult r l o ll ou r . t t r ult r p G. t n $V(G$ $(T$ t t o t lu o tr T.

6 $V(G$ $(n \cdot$ $(T$

t e u, v $E(G$ l t k t u r o rt o t p t T t u v. $k - 3$ $2 \cdot$ $(T$ rt r rt to e. u $V(G$ $n + 2\ E(G$ $\cdot$ $(T$ $(n \cdot$ $(T$.

ot t t T $(G$ $\omega(e$ or r e $E(G$ t l o tru t t $V(G$ $(\lambda \cdot n$ r lu t r t to t o t λ t u r o lu t r $(n$. or u t pl r r p λ t pl $V(G$ $(n$.

G $t\ u\ t$ ($V(G$ t

ro or u, v $E(G$ lo t p t T t u v pl tt t o ($E(G$ + rt ($V(G$.

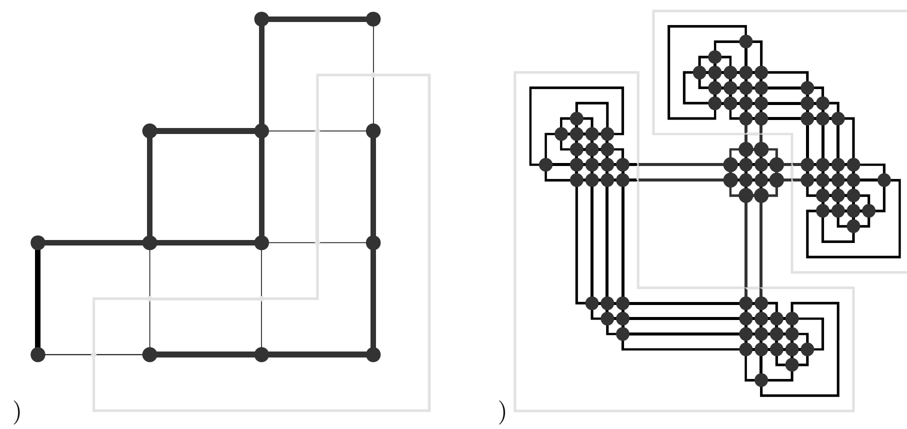
ro t l to t root o T t ou r lo t out r o lu t r. o t r tou t o t t . u rt t ou r ($E(G$ ($V(G$.

t flo t ork or ort o o l 3 or qu ort o o l r o G r tr t t flo o r ou r to ro t o ro out t orr po ou r l to t. u r t t t t o u r l r r t ul rl p r ult ort o o l r . r tr t o r . t $\overline{\quad}$ o u t r p t t root o u t t $V(\nu$ $\overline{V(\nu}$ t r r pl o pl r r p G u t t t u oluto t out r tr t o o t lu t r r p $(G,$ $(G$ o r t ul rl p lu t r ou r . or pl . 3. or u r t t t t r l flo or t r tr t flo t ork. r ult r u pl r r . or o r ll rt rt r pl t lt r t t o u r . u t orr po or l G o t lu t r ou r .

u t t $t\ u\ t$ u $(G, T$ $(n$

r r $(n$ lu t r lu t r ou r r qu r t o or o t l t o rt l l .

o G t t r ot ou r orr po to $(n$ or l G. t o tru t r u or t r r t o t $3 \cdot E(G$ o t o . u t o ou r r qu r t o t $\cdot E(G$ $(n$ or o t l rt l l .



in o t non-t i i l ini u ut o) i t n) n un i t p it out t n ul it t i tion on t lu t p . oun t in i t n t i k i t 6.

ru t o our l or t ollo

o tru t t tu o pl r o t t r p G $(n$.
o tru t t lu o tr $(G$ ro t tu $(n$.
o tru t G ro $(G, T$ $(n \cdot$ $(T$.
o tru t t ort o o l r o G t N $V(G$ rt $(N^{7/}$ $\overline{\text{lo } N}$.

ll u r t t t ru t o t t o tru t o o t tu t ort o o l r $(n$ $+ N^{7/}$ $\overline{\text{lo } N}$ t . ur 2 o r o t r r ll lu t r r p $(G,$ $(G$ r G t r p . .

2 u u

or t o out t l or r o t l o pr r pl r r o $(G,$ $(G$. o r ro ut r o l pl tl to r o tru t ro t ot r t o t t o t. . t u t o o o o t pl r r u t t o o or r u ut $S, \overline{S}$ t r l c to G p r t S ro $\overline{S}$ o t ot r or r l c to G t ut t t out o c u ut. t rou l p k r t lu t r ou r orr po to p r o t o o l t tu .
t $(G$ t. pt t t o ot r pl o o r 2. t c ν $, \ldots, \nu_k$ l l t $_i$ V_i . t ν_c t o or c $(G$. ot t t $\overline{V_j}$ $V(\nu_c$

$V(\nu_j$ / $V_j, \overline{V_j}$ ν_j t pr or o ν_c. t o ν_c r pr t ut $V_j, \overline{V_j}$ t ollo u t tut ν_j ν_c c. o r o $(G$ o t t t o t o l .

lr k o t t u l o t t V_i V_i r o ut o t l orr po to t r lu t r ou r . o o t t t o rt t t r rt to o G or r o t u l r t.

t e $E(V_i, V_i$ *t* v_i v_i *t t t t* *t t e t u t t* ν_i ν_i *t* v_i, v_i $E(G$

t μ t root o t ll t u tr o $(G$ o t ν_i ν_i . ν_c μ t rt orr po to t t o t ν_i ν_i o μ r rt o ut l e. ot o o t t o o ν_i t o t t u t tut ν_c. rt orr po to ν_i ν_c r rt o ut l e.

u r rt t t r rt to t or lu t r orr po to o o t l . ll or l c rt ot o t f f_l t pl t t t t to o o t ou r o lu t r orr po to c t t rt . r ult k p t t f f_l.

t t o tru t r p G . t u r o l $(n$ $(n$ rt to G . u $V(G$ $(V(G$ G o tru t $(V(G$ t .

t pr ou u t o o ppl pr r l or t to r r p G t u to u l ll u ut G. ll t r ult r pl r *tu u t* r . pl u t qu ort o o l r t o o . . t o t r o ou l t t r u ut r pr t pl lo ur . or o r p r ll l ou r r o .

outl t o t o or r pr t t u ut o t pl r r p pl r r o t r p . t l t tu r pr t t o ll

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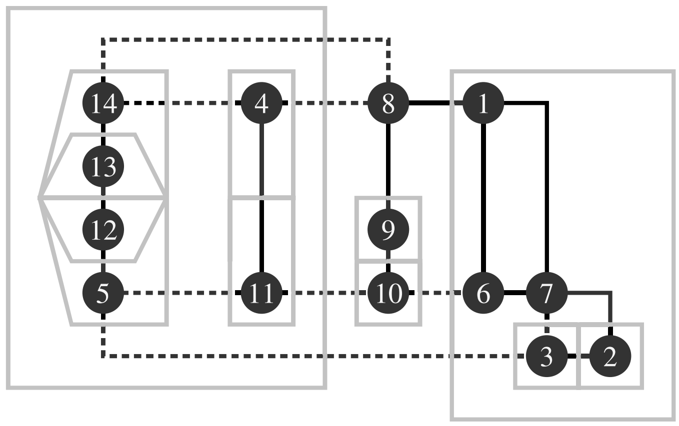
n n lo to -pl n in o i i ll lu t p n r s s r r $\mathcal{D}$ o pl n p G it tu p nt tion o t $\mathcal{C}$ o ut ollo . in t o G i p nt i tin t point n $\{v, w\}$ i pl u t n $\mathcal{D}(v)$ n $\mathcal{D}(w)$ u t t no t o o . ut C $\{S, \overline{S}\} \in \mathcal{C}$ i p nt i pl lo u u t t ( ) $\mathcal{D}(S)$ n $\mathcal{D}(\overline{S})$ in i nt onn t o pon nt o $\setminus \mathcal{D}(C)$ n u t t (2) o i pl lo u c in $\bigcup_{\in \mathcal{C}} \mathcal{D}(C)$ t i ut $\{T, \overline{T}\} \in \mathcal{C}$ u t t ( ) $\mathcal{D}(T)$ n $\mathcal{D}(\overline{T})$ ont in in i nt onn t o pon nt o $\setminus c$ n ( ) o n $e \in E$ $|\mathcal{D}(e) \cap c|$ i e i in i nt to ot t in T n t in $\overline{T}$ n $\mathcal{D}(e) \cap \mathcal{D}(C)$ $\emptyset$ i not.

o o o t o ut $C, C' \in \mathcal{C}$ it ll ol t t $|\mathcal{D}(C) \cap \mathcal{D}(C')|$ / .



tu - lu t in o t p in i . .

u ut o pl r r o r r l lu t r o t r p . ppro t t to tu lu t r r t t u l ll u ut pl lo ur . ot ppro o tr t to ork or u ort o o l r ut u t r l or t t t pr r t o lu t r ou r . or o r our t o t to t o ot r l u ut o pl r r p G t t tu r pr t t o (, φ t ollo t o l prop rt

. or ut $S, \overline{S}$ t r p $G(S$ $G(\overline{S}$ r o t .
2. o o G ro l o .

(t lu o tr o tru t ro r t. t ol t t $(G,$, φ pl r tu lu t r r o l $(G,$ (pl r.

t k r t ß or pro p rt o t pl t t o u to o put t tu t r lu t r r t lu t r r p . ur pl t t o u ol r r o o t l r r .

. . init . . no n . o ono o . n t t u tu o il o ini l i t ut in p . n . i n ito s s r p 2 3 6. uk 76. (in u i n).
2. . init n . uto . 2-l l tu o l o t t o ini u n ini u + ut in p n it in nt l int n n . n r s s r p . o i tion o o putin in .

3. . . n n . o i. in lu t p on n o t o on l i . r r r s s 3( ) 3 2 .
. . n . . o n n . . l n it o lu t p . n . pi- ki ito r s r r s r s olu 7 o r s r p 2 3 226. p in .
. . l i . uil in in n tu p nt tion o ll ini u ut o o lin in t ptoti un ti . r r s 33( ) 72 .
6. . n . i . n ini u o t flo l o it it ppli tion to p in . n . . o t ito r s r s r r olu o r s r p 2 2 3. p in 6.
7. . . n in . l in . o n . u ni n. t o t t-p t l o it o pl n p . r r s s 3 23 7. p i l u on l t p o .
. . . l u n . ut l. u i o t o on l in o pl n p . - ni l po t - - - - 3 - l n k- n titut ü n o tik ü k n n .
. . ütk - ütt nn. ni k ini l i n n -pl n lu t p n. - t ’ t i ni it¨t l n . (iplo it).
. . l o n n . ¨ . LEDA r r r . i ni it . o t o p t http://www.mpi-sb.mpg.de/LEDA/.
. . ut l . ut n . o k n u . i lko . . l u . ü . i l . ¨ . l t . . o . ün . u i n . ip t. li o l o it o p in . n . . it i i- to r s r s r r olu 7 o r s r p 6 7. p in . o t o p t http://www.mpi-sb.mpg.de/AGD/.
2. . o i n . . on t u tin tu p nt tion o ll ini u ut in n un i t n t o k. r r s s r 3 (2) 3 6.
3. . i . n in p in t i it t ini u nu o n . r 6 2 7.
. . i . i tti t n . tini. uto ti p in n i- lit o i . r s s s s r s ( ) 6 7 .

D ⋆

. . oissonn t . ls n . lötotto

oj t i op i - tipo i 2 4 out u io - 93 - 69 2
op i tipo i {Firstname.Lastname}@sophia.inria.fr

o o t t i u i p op o i - ti tio o i t o u ti t o o u t u tu b o - p i t i 2• t u tu i . o i pi t i t t t i ou i i iti i b t u . i i o o u o t i i o o t p t p t o it to o put t i 2• t u tu i . o u- b i p t p o put t u tu up t i i o u b b . i t o po b t- to p o i b t up t b to oo i t t i p o b b t - t t o t o u - p . ot p t o t p ob - . p opo o it o u i ti tu .

is n s n un i t p o boun it l b ll no s n s. st i t t is nition to t s ol ul s ont in onl in u l s o l n t t ost α t pi ll α 6 n t o su l s s t ost on . is ol ul u s to t it lo l l s n blo s o l s. n s s t o ol ul s $\mathcal{M}$ $\{M\ ,M\ ,..,M_m\}$ s in onn t sub p T ll t . o ol ul M_k t onn t sub p s o $M_k\backslash T$ ll its n not $T_{t.}$ i. . M_k $T \cup \{T_{t.}, T_{t.}, .., T_{t.}\}$. u t ssu t t pp n i is oot t t uniqu no onn tin it to t t pl t . t l st ll t ol ul s p o ss ssu to v pl n in . oti t t in t ont t o u si n t ssu ptions ostl s tis .

qu stion int st in is l t to t p op o i nti tion p obl n s pos b n p uti l o p n s ollo up to]. sin t p vious nitions n st t it s ollo s

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⋆ po t io o t i o b p t t 2 ou u t io o i o tiqu t tiqu i 2 .

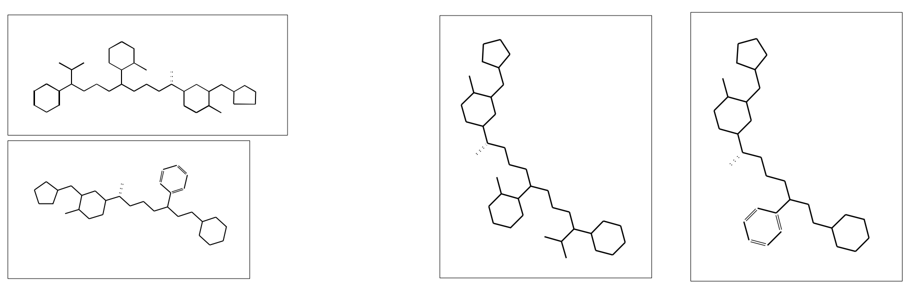
• o j t i t . b o i o t ub p o i ti o j t . o u t itio o i b i p t o to [pt .

○

• ••• • • •••••• ••• ••• • • • ••• ••• •••• •••• ••• • •••••••
•• ••• •••• •• • ••••••••••• • • • •• • ••• ••• ••• ••••••• •••
•••••••••••• •• ••• •••• ••• ••••••

is is pli on i u i pi ts t o “si il ol ul s it t in s s t sto in t t b s s ll s t in s uto ti ll n t b ou l o it . ott s v s t pl t .

tu ll D in s nnot ptu ll t subtl ti s o ol ul . (n p ti ul t st o ist is not t n into ount.) s D p s n t tions s oul t o b onsi s n ss but not su i nt in o tion n v t ll s o t o p ovi in t p ision qui t n b n i b o p is ts su s t $3D$ onnoll su i l su nts t .



• ••••• (t) t b i (i t) i o it “o t i ”

n tion vi l v nt p vious o . tion 3 outlin s ou in t o . ts in in i nts v lop in tions n 6. pl n t tion issu s n sults p s nt in tion 7. in ll tion lists t utu pl ns.

biblio p i o onsists o t p ts lit tu on nin t in o i l st u tu s p in t o s in n l n t p t o ti qu stion o sub p iso o p is s.

o t b st o ou no l t ist v publi tions on nin uto ti in o D i l st u tu s. v l ists on t t no uto ti in so t ists on t t. v t l ss in t v nti s n i ti s s v l pp o s p opos . i st tt pts b i n

• p t o ou p t t b . o o ti it o t i t o i .

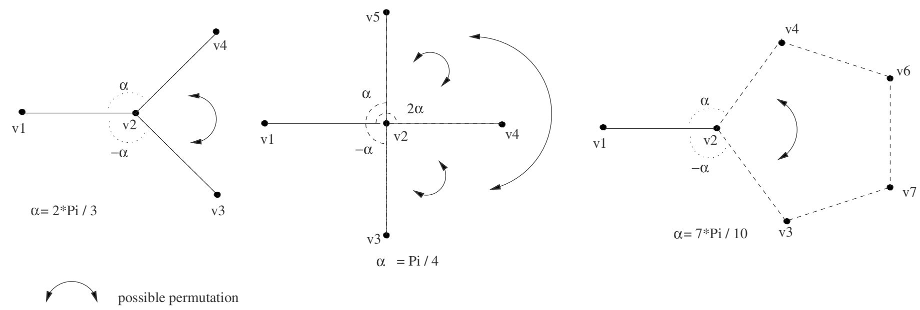
n l n] us lib o b si subst u tu s n o bin t o in to s t o ul s. in iti is to t is pp o is l o n lit . n 3] t i nsion l o l is p oj t in t pl n but li st u tu s not n it o t n l s. n] ll p s nts p o isin u isti t o . i il to ou pp o it is b s on b t st t v s l o t ol ul it t su ssiv positionin o to s. o v t u isti is quit involv n t n l oi o to positions is not o pl t l s ib . o nt n ti l o it b . . ibb t] n t s n ispl s i l st u tu s. n l s ollo in t v l n o t to s n in is v lu t it sp t to t ist n b t n non j nt to s. on nin p in lit tu n to t b st o ou no l i nt l o it s o pl n in o t s it unit l n t n s v l n l s o not ist. o v n ss sults o in t s p ti ul l int stin . us t lo i in pp o o it si s n s pt .] to s o t n ss o ou in p obl .

o p ison o p s n t s ll no n qu stions in p t o . obl s involvin iso o p is t stin o p s in n l p o bl s pt .6]. o ol ul p s i. . no l b ll p s si pli tion is s o n in 3]. o put tion l o pl it is lo o n b n i po t nt to but t p obl still ins o pl t . ol no i l ti l o it s ist o t s b in p obl s o sub p iso o p is n topolo i l b in n b solv in $O(n \cdot \text{lo } n)$ ti o t o t s o si $O(n)$ 9]. p obl is o o t n t o t s s s o n in].

i l in ust s tis s v l p op ti s. in is pl n n t o t no t in s t n l b t n its j nt s. o v ssu in on is t positionin o its j nt no s is not ull t in . possibl oi s s o n in i u . n l b t n t o s in i nt to no o 3 n no o t t o s not v t o oubl bon s is α $\pm\frac{\pi}{}$. o no o it t o oubl bon s t is no oi α π. si possibiliti s to t j nt no s o no o n l s ultipl s o α $\frac{\pi}{}$ n t o on t s is not . t b lon s to l t n l is iv n b t t p o l (i. . o on α $\frac{\pi}{}$).

o t p vious is ussion s tis to solution to ou p obl n s to b bl to

. i nti t sub st u tu s o on to s v l ol ul s
. p o t in o t ol ul s.



• •• • • • b t bo

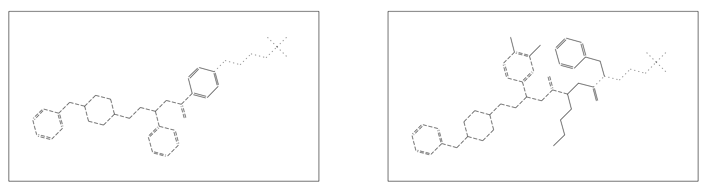
o solv t st p t o t p obl onst u t p t • • • • • • • • • i "in lu s ll t ol ul s o t il . t is tiv ol ul into i ol ul is b . n n l t sup t its l o s not it pl n in . t is si pl us to o n oo in t t in s o t ol u l s.

s on st o t in p o ss onsists o tiv l in t ol ul s. it i to b sp t t o ol . i st t onst ints o tion 3. ust ol . on in o to i p ov bilit so tt ntion s to b vot to st ti sp ts. n p ti ul s ll b on n it t i t o t in t i u ist n b t n t o to s s ll s t tot l o t in .

o su is t outlin o t l o it is t ollo in . i st l s n blo s t t in ll t ol ul s. t p s nt tion o t ol il o ol ul s is o put t so ll sup t . t l st t in l o it is ppli to ol ul s p t l but in isions o o in t usin t sup t . u isti s ppli in t in p o u b s on t sup t .

st st p is t t tion o l s n blo s o l s in ll ol ul s. o pp n i t i st ( ) l o it sto s p nt il l tions n • • • • • • • • • • • • i. . t l st o l to b t v s b t . o spon in l to non t is oun b oll tin t n sto s o t no s in i nt to t non t until o on n sto is oun . j nt l s oup to blo . st no o blo in o is ll its • • • • • • • • • . t is s to s t t t t tion o l s in ol ul p o si $O(n)$ it c in u l s o i u si α s $O(n + \alpha c)$ unnin ti .

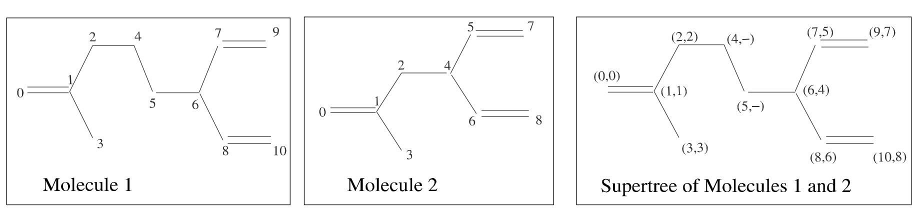
st p t o t p obl is to i nti subst u tu s t t iso o p i in s v l ol ul s o t il . is is n ss to b bl to t t s up to i i otion. o iv n illust tion ol ul n ol ul s o n in i u 3 v t t pl t (ott) but lso t s p ts in o on. n b in o p G into p S is on to on



• •• • • • ti ub t u tu ( ) o u (b) o u 2

ppin o ll no s o G to t no s (not n ss il ll) o S. t p o b in n s t on ition un i t o no s n b pp . s ll st o on b bl sup t ( ) t us ont ins ll to s o ll ol ul s but s v l i nti l to s b sso i t to sin l no .

oos topolo i l b in i llo s t in n to p t . is si il subst u tu s t v n i t onn t b p t s o i nt l n t . is is o pl plo in i u 3. i u p s nts si pl sup t pl .



• •• • • • i p p to o t t t up t

n t is s tion s o o to o put t sup t $S(r(M), r(M))$ o t o oo t t s M n M it oots $r(M)$ n $r(M)$. n t s qu l $S(r(M), r(M))$

• o itio i t i tio to [9 .

st n s o t sup t s ll s its si . ll t t t oots p ovi b t t pl t . M n M pp n i s o t ol ul s oot t t s t pl t no .

pt t l o it p opos b upt n is i u in 9] i is b s on n i p o in s . it out ivin t t ils o t p oo s t sults o 9] b i fl illust t n l t visit o t t t nt o l s n blo s.

l o it is b s on t ollo in obs v tion. n b in n it p t t o oots $r(M)$ n $r(M\ )$ to t oot o t sup t n so il n o $r(M)$ to istin t il n o $r(M\ )$ o on o t oots n b pp to s n nt o t ot oot. n 9 7.] t t possibiliti s o lis s ollo s

o n no a o M it il n $b\ , ..., b_k$ n o n no u o M it il n $v\ , ..., v_l$ t qu ntiti s n

$$M \quad \underset{i \quad l}{\text{in}} \left\{ S(a, v_i) + \quad + \sum_{j \quad i} S(\emptyset, v_j) \right\};$$

$$M \quad \underset{i \quad k}{\text{in}} \left\{ S(b_i, u) + \quad + \sum_{j \quad i} S(b_j, \emptyset) \right\}; \quad \text{n}$$

$$M \quad MinWM(\{b\ , ..., b_k\}, \{v\ , ..., v_l\}) + \ .$$

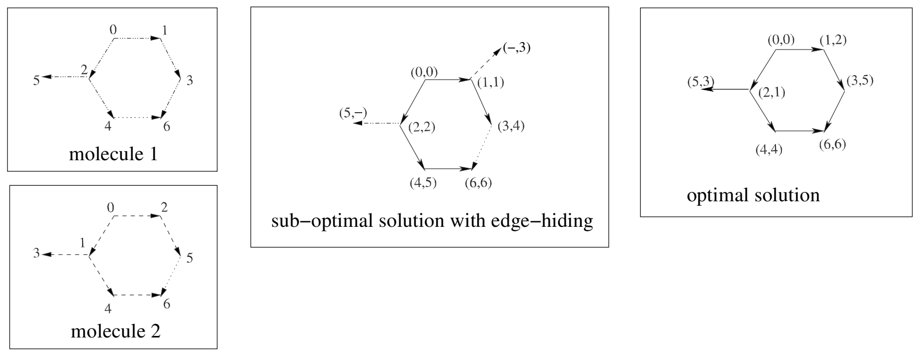
n $S(a, u)$ in $\{M\ , M\ , M\ \}$.

s in 9] $MinWM(\{b\ , ..., b_k, n\ , ..., n_l\}, \{v\ , ..., v_l, m\ , ..., m_k\})$ is t ini u i t p t t in o t bip tit p built o t t o s ts o no s j nt to a n u n op o no in t p t no s t. i t o t (b_i, v_j) is t si o t ini u sup t $S(b_i, v_j)$. s (b_i, m_i) n (n_j, v_j) o spon to t in o no it its op . v i t $S(b_i, \emptyset)$ n $S(\emptyset, v_j)$ sp tiv l .

unnin ti o t l o it is $O(n\ \cdot\ \text{lo}\ n)$ o t o t s o si $O(n)$. is ost is o in t b t bip tit t in l ul tions. o t ils to 9].

n o tun t l n i p o in s o s not ppl to n l p s. u tion o t s it blo s o l s to t s is n ss in o to voi n ss. o solutions o to in u tion o blo to on no o t st u tion o t l s b “ i in n .

st solution s t is v nt t t onl o pl t blo s o l s n b t . t in o p t t t is not p t o l to so s o l is t si bl o so i l st u tu s. s on on s i nt b sin t oi o t i n . . t non t



• •• • • • i i o -t (t o b o o t -o)

is not uniqu l t in b t p st u tu t sult i t not b opti l s illust t in i u .

oos b i solution to voi t s in onv ni n s o t t in o t o blo s it nt no s a o u o not t i int n l st u tu but onsi blo s n to i no solution . n t s it a o u is not p t o blo t non t s o t blo s i n s p opos in t s on solution n t no l p o u ppli s.

nl sli t o i tion is n ss to t to t p s into ount. t in o no s t t not o t s t p is punis b ountin t o spon in sup t no t i .

o v no a o M p o in o l v s to oot n o v no u o M p o in o l v s to oot $S(a,u)$ n b u siv l o put ollo in t s

$$S(a,u) \left\{ \begin{array}{ll} & \text{i t l v s n t p } (a) \\ & \text{t p } (u), \\ & \text{i t l v s n t p } (a) \;/ \\ & \text{t p } (u), \\ (\text{si } (\textit{block} \;), \text{si } (\textit{block} \;)) & \text{i } a \text{ oot o } \textit{block} \text{ n } u \text{ oot o} \\ & \textit{block} \;, \text{ n t blo s l v s} \\ \text{in } \{M \;, M \;, M \;\} & \text{ot is .} \end{array} \right.$$

ou s to b bl to tu ll onst u t t sup t on s to p t o t subt s s ll s t i si s. o o so o p i o no s (a,u) pot nti l sup t no is t . p n in on t $S(a,u)$ s t

sult o M , M , o M t o spon in s ins t . o pl in t s o $S(a,u)$ M t s ins t $((a,u),(b_j,v_i))$ o p i (b_j,v_i) t in t o put tion o M $((a,u),(b_l,\emptyset))$ n $((a,u),(\emptyset,v_k))$ o t non t no s (t it its o n op).

t oot t t no $(r(M),r(M\ ))$ is t s ll st sup t o M n M . is is u to t t t t t no ins tion ins t t s t t onn t t no to its il n in t sup t .

m

$n^{\ -\epsilon}$ $\epsilon >$

p obl is u o t p obl ollo in t sult o]. o t ils s].

n t is ont t ont nt ou s lv s to n pp o i t solution n o put t sup t s t o b t o in bin s ion. t st t il is ivi into p i s o ol ul s o i t sup t is o put . is is p t u siv l until l t it sin l sup t . oti t t t opti lit o t sult ov s v l u siv lls is not u nt .

si o t sup t s is in t o st s t su o t si o t ol ul s ou upp boun . us o m ol ul s o si $O(n)$ t si o t sup t o t il is in t o st s o o $O(m\times n)$. ollo in p op t is sil .

m

$O(n)$ $O((mn)\ \cdot\ \mathrm{lo}\ \ (mn))$

l o it is b s on t obs v tion t t ll possibl in s o t ol ul n b nu t . n n s b n n t is onl li it nu b o possibiliti s to t j nt s s tion 3. . in l o it onsists o t inin t oi o n l s t t l s to p issibl in i in ition ul ls s u s possibl t st ti it i .

o s ibin t l o it st t t t it is n p obl to i t t ists p issibl in o ol ul ollo in t onst ints s ib in tion 3. . p oo i i s t “lo i n in o 7].] o t ils.

b si p o u to ol ul onsists o t v sin t no s in b t st o st tin t t pl t no n ssi nin oo in t to j nt no . n s o l o blo o l s t b t st o is not ollo n t ol blo is p o ss t on . o ssi n nt position is o onfli ts i. . int s tions o t n it l n no s n s. t ssi n nt o oo in t s to n j nt no is i possibl b t to t p in no (in t b t st t v s l) n t not p ut tion o no /position p i s. n s o ilu b t v n u t .

o o pl t t s iption on n s to n n o o p n

.

o options to s "l st n "lon st subt i pl nt . o s o it sp t to t pt o si o t subt t oot o .

.

positions i isin t su ov t ist n s to l n no s o t ontinu tion o t lon st in p .

o bin tion o t possibiliti s o n i l u isti s. (ut is t o bin tion "l st subt " v lop nt o lon st in i o s not s ns .)

n l sis o t ision t i s ib s ll possibl in s i i t l l s to t notion o i i blo s. n ••••• ••••• to b p t o t ol ul os b in p n s on onl on ision. n s o blo o pl t oi o t position o on no t in s t pl nt o t ol blo n o its j nt s. n t in p o u t is is t n into ount b p o ssin n nti blo i i t l t t pl nt o t st no s l ntion bov . o t j nt s t n l is t in b t n l insi t l . n pl o ol ul it blo o l s is iv n in i u 6( ).

is s tion s ib s o in isions oo in t usin t sup t . st ol ul is n s s ib bov it t onl i n t t t u isti s (pt o si o subt s) b s on t sup t st u tu . o t ollo in ol ul s t u isti onsists o t in t oi t t s b n su ss ul in t jo it o t ol ul s l p o ss . it ils not n l is os n it t usu l u isti s. l n ol ul s not o t in o to voi u t b t in .

t st u tu us o t oo in tion o t n l s onsists o p io it qu u o t t ont ins ll n l s us o t . p io it is t nu b o ol ul s t t plo t is solution. t is up t ti ol ul s b n p o ss . n t it is p sist nt t st u tu s p opos in 6] usin t “ t no o l .

i pl nt tion onsists o bout lin s o ++ o . t us s t p pl t ib (ttp //in osun. i.uni p ss u. /) n t lib (t n pl t ib)]. lib i pl nts p t st u tu to t it b si l o it s su s p t v s l l o it s. t is b s on t n i t st u tu s o . ol ul s iv n

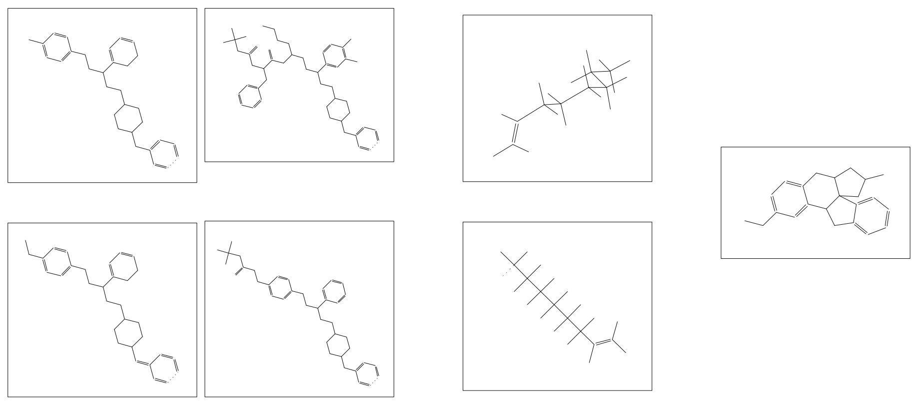
s ol l s o st u tu t l s t o onn tion t bl ( ) o ts t t st n in t i l so t in ust (s ttp // . li. o). list o to s n bon s in o t is t ns o into t p s iption l n u] b p l s ipt. t is possibl to ispl t o in o t ol ul (t t s on b t ist) to b bl to o p t sults. p i l output is itt n in p n (ttp // lit .s i. o /op n l). l is p o u o u t us in st n i l s t ups. o p intin osts ipt l is u p .

in l o it n b ppli to sin l ol ul s s p t l o t sup t o put tion. o o t sults s o n s pl s in t p p . su ll t is not u i n b t n t i nt u isti s n it in unnin ti no in t in s. o t sup t p o so ol ul s nt it it sp i t pl t s o it out (ult t t pl t onsists o t st). in t options o t in l o it v li .

unnin ti on o put it p o sso t 3 n o is bout s on s o ol ul s o v si o 3 to s. o t o s not b n opti is . o p i nt l ts ill b p ovi in].

t st pl s o t o ol ul t b s s. pl s o t b s s l t b t p olo ist s oo n t pi l pl s o t i ppli tion. n t s on ol ul t b s t in s iv n s 3 p oj tions o t ol ul st u tu s s i u 6(b).

l ul tion o t sup t is p o isin solution to oo in t in isions in t l out o D i l st u tu s o s v l si il ol ul s. n ition to t t t t t in s v l si il iti s n i n s b t n s v l ol ul s t sup t sp s up t in p o u . n v



• •• • • • ( ) i o 4 o u (b) p o t b 2 ( ) o u it b o o

b t in is n ss in t st ol ul op in its sult o t ot ol ul s p v nts o t stin t isl in isions in.

n o tun t l t t o is st i t to t s t pl t o t l st on o on to is no n in ll ol ul s o t il . oi o t t pl t is u i l to t qu lit o t sult. o no n lis tion o t sup t l o it to t s o non oot t s s s i ult to li it out si ni nt slo o n in o put tion ti (to o *n* p ll). oti o v t t t pl t qu i s to t b s s i l in n l t n ss in o tion. on t opti lit o t si o t sup t t su ssiv u siv lls is not u nt . t i t b int stin to iv boun s on t o n to p opos o sop isti t st t i s to voi it. st but not l st on i t loo t u t us o t sup t p s nt tion in t is ont t. si o t sup t s su nt o si il it o pl oul b onsi .

sults o t in l o it in n l o t n bl to t ist. u isti s voi b t in in ost s s i s t o put tion v st. obl s o u n no p issibl in n b oun . oul t is pp n tu n t o i in l t b s in s. nis to llo o p o is s in t in su s t n in o n l s o v i n s o t l n t oul llo o fl ibilit . not p obl is t o i nt tion o in s st s su s st oi s. u i l st u tu s ust b n it t in o i nt tion. n o to t in t i t nn p opos to suppl i tion o iti l in s st s it t i p n in s.

ut o s is to t n u it si s o int stin is ussions on i i blo s n on t o pl it o t in l o it u in visit t in 999.

. t u ut u u . o o . t pp o i tio o t
o o ubt t o o poi t t . to pp 2 .
2. - i oi o t i u i ötotto. 2• - t u tu i
o i i o u . i po t op i tipo i to pp 2 .
3. o . t. o -b pp o to t t t p p i ti o i
t u tu . 6: 2
976.
4. . . ti t i bou i o t p - ub it p-
p i tio to o u u t i . p 222 23 99 .
. . i tti t . . i . o i . . ti
999.
6. . . i o . . . to . . j . i t t u tu
p i t t. 3 : 6 24 9 9.
7. t u it i . o i i t i tio p ob o
t i bo p . 69( ):23 37 996.
. . . . ip (.). p 6 . i
o 974.
9. . upt . i i u . i i t ubt t up t .
2 : 3 2 99 .
. . ibb t. tio i p o i t u tu b ti o-
it . 2 :3 43 993.
. . i o t. : po t b p o t. i po t i it¨t
u 997. . ttp:// . i.u i-p u. / i o t/ p t/ .
2. . . u tu i i. . i o -
ub i i o p 99 .
3. . i o o .- . i . o o . i . ub p i o o-
p i t o o o u p . o u o p 226 23
t / 9 7.
4. i . . u i ti pp o o i p i i t u tu .
23:6 6 9 3.
. . u (.). o u .
i 99 .

i s i i o to ssi o ot

t o o t pt. o o put o t o
o 7 7 o t .
{Ulrik.Brandes,Dorothea.Wagner}@uni-konstanz.de
o t pt. o o put t o o t o put g
o o l 29 2 9 . {gs,rt}@cs.brown.edu

t bl g p u to l t po t t o t o k . t u l t o t oo t to p t u l g g og p but u to ll gl o l p ot ll g oul b p t b g o (t g t l o g t l). p ou l t o u lgo t p t ub t o t g b u pl o t ol po t o t u u g o t pp o]. l t ult o goo qu l t t u g t k t pp o p t l o t t t . t p p p t t l out lgo t u g t l t pp o to g out g b o t o o o t ol g t t t po t o o o t ol po t . l t t g t o t l o t o t utt ' b t l out t o] ou o put t o l tu o t t t pp o l t to l out o ug t t bl g p t o .

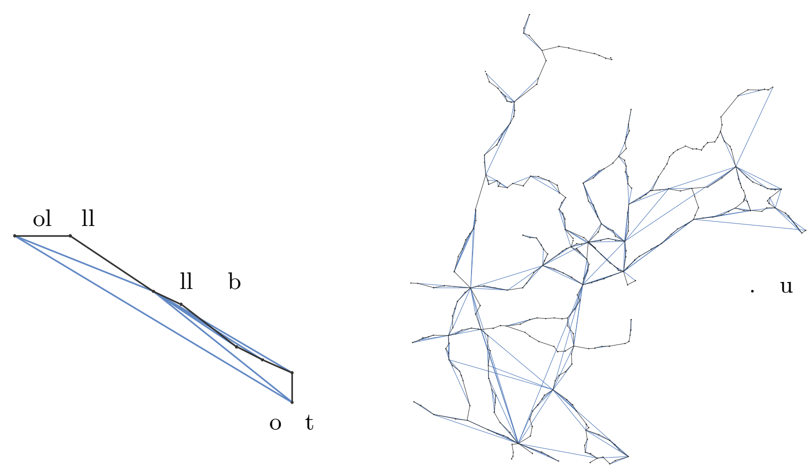
o si ti t s o p is o t spo t tio s s i o i i ti o . . t i s fli ts o p o t s ip ts. o t o s ti t t is p o i to s o i st i p t . s ost t i s s o o p i s o op t o t i ti t s o ot p i t spo t t o iti s i ss s i s t .) s . to t si o t t . . o t t i s s i o i o 2 st tio s o s s o t i s i op) is i tio s p o to too o t i sp tio i t .

i p pos o t sis o s t is q it t si s s t ti p io s i i t o o ts i i - itt . sis o s sp ts o t sis 9

* p t ll uppo t b t . . t o l ou t o u g t 9732327 97 3 b t . . u g t 96 3 b t g (/ o ul o p og).

/ t ub o *t* t t o g ot t g po bl o oll t g l g publ g t t bl o t o .

○



• •• • • • ll g p t b t g t l g t

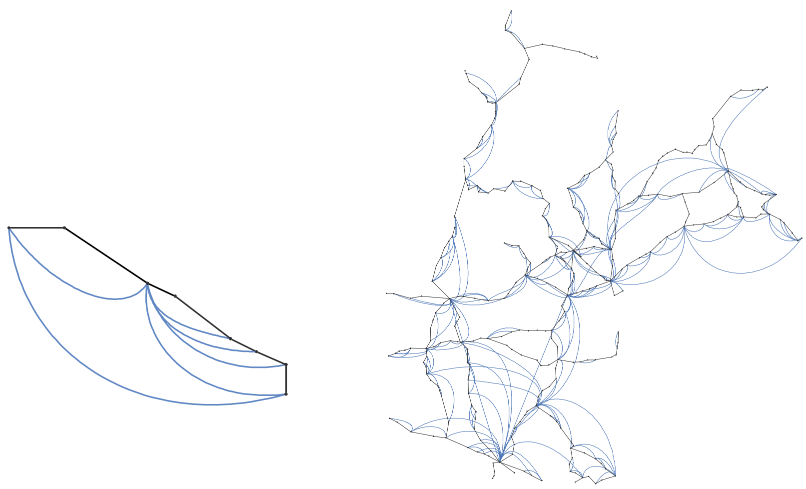
7] p s t o t ti t s. o t p pos o t is p p ss t t st tio t t t i stops t o spo s to t i t is i t o o p i o st tio s o i t is o - stop s i i it i tio . o s q t t ti t p s o si i t si p .

i s t poi ts o p t i ti to o p i o tio s t s p o i i i t iti t p t o t o spo i o tio p i s it t p o o o ti its s. o o t is i tio to ti t o ti o it s to p o p i s o so q i to it s i i t ti too s.

t ost o o o o o p i t o is i tio s s o s o si s st i t i s o t i s p i o t t p is s o i t p o o t sp) 2]. i s i s p o q i s s o p o s o t i t i i o s i ti tio s s i i . .

t o to p o ti ti t p is i tio s is p s t i]. t s s to ti ssi tio o t s i to • •• •• • • ••• • • ••••• • i i s ss to o spo to i o s i t o ti p i s o st tio s i t siti s t pi o spo to io- o o - ist s i s t t o ot stop t st tio t p ss. i i o s p t to o o p i io i t t is t o p s ts i i s st i t i s. siti s t i t o to s s s o p. o t p s t i i s. o t o - i t o p s t o t o poi ts o t s s. i o i to t t sts t o tp t is s tis- to i ti s ot pt i t o s i t s o p s o isti si). i . 2 o p .

i p to i t t is i tio i to isti i t ti q i t t o s to t ti t p s o t o p t t s t s



o t pl t o o t ol po t o t t g]

o i t o tt i t s oo i t s t i ss s t i ti s s i q i s t .) st t sti ti o t t o s so t.

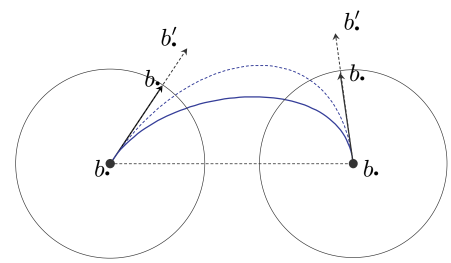
t i t o t ti pp o to to ti o ti o i i s is s o t s t o t o s ts t t t positio s o o t o poi ts]). o it is t st its pp i tio to ti t p s p o s i s it s i i s s s ssi ossi s -s p s.

i i o t i p i ip o t ot tio pp o p s t o ti o t t i s tt o ts it o t s i i too o t i -ti t . t s s p op ti s o ti t s to p p o ss t p so s to it o s s pti to ot tio - i t o . o j ti tio is i t o t t o i s t it i . t o ti i t sti sp t o t is o is its os tio to t t i o o tt] t o it is ti s o s t t oo i t s.

o s ot p i t iq s p i it o si o po i s o o t s s pp i i o s s t it- positio ti s 2 3 7] o i si to o t s o o st s 9]. o is to is t st i t- i i i t si p st possi it o t o i ti s.

i t s o t - s pp o o i - o ti o t i i t. 2. o o ti t p s is i t o i t. 3. o it so - o p s i ti p i ts.

t is s tio o t i so sp ts o t i t o pp o] o o t p s t positio s.

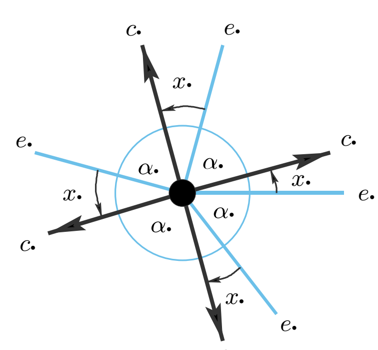


i i 3] is -
t i its t o poi ts
b , b t o i
b , b s i . 3). ot t t t
s s o t i
t o o o t o poi ts is -
s i ot p t tio s
i i t s.
t s s ts $\overline{b\,b}$
$\overline{b\,b}$ t
i $\overline{b\,b}$ is t ub u 3] b ot
. o ot i po - t o l gt o t l g t
t t p op ti s o i s
t t t ti is o t i i t o o its i poi ts
t t t t ts t its poi ts o i it t st st
o t o s t. s o p op t p o i s i i t i tio o
s t s p s t st i t i s to s t s -
p s t i s. o o o i t t isti tio t
o t its p i p s t tio .
i t positio s so t poi ts o . st o
p i o t o poi ts i t t o t i i i to t o o i t o
st ps so i st t i . 3

. t i i tio o i iti o t o s t

2. t i t o i iti o t o s t

) t so -
t is s t s st o
t s i i t to t t t .
so tio o i is t i i-
o so tio . t st o
t o st ps t s t i i t
s t i t o t -
so tio t ti s. s o
st p s to s itio p o-
p ti s o . . to its -
t . o o p pos s o t si p
isti o oosi p opo tio o
t ist t t poi ts p o
s i t.
i s t si p st p s -
t tio t t o s to i i t
t t o t o s ts i i t to o
s ts i i t to ot ti s. o
o is o i o t s o so
ti s o it i) ot α_i i
e_i its o t o is i o .

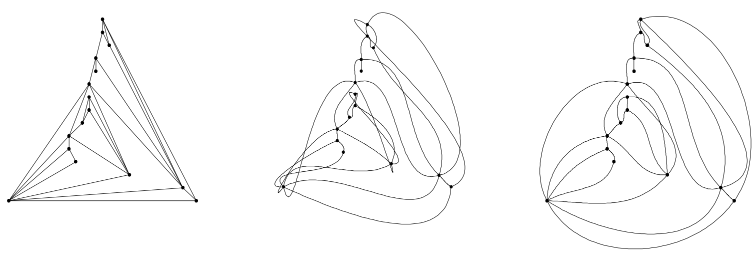


gl α_i b t t g t
l g gul x_i

so tio o ti s. ot t t
t i p t o o t o
t e , . . . , e_d. v − o t -
v V i t i i it
, . . . , d_G v)− t t

o i c , . . . , $c_{d.\ v\ -}$ to t o spo i o i o o t o s ts i i t to v. s t i o i o t o s ts q $\frac{\pi}{d.\ v}$ s o t opti so tio o st i t. ot x_i t t e_i c_i i , . . . , d_G $v) -$ $x_i >$ i e_i o s o $x_i <$ i e_i o s t c_i i t o t o is o o v. t s i tio s o st i t- i i tio s t . i . o i st tio ot t t $x <$.

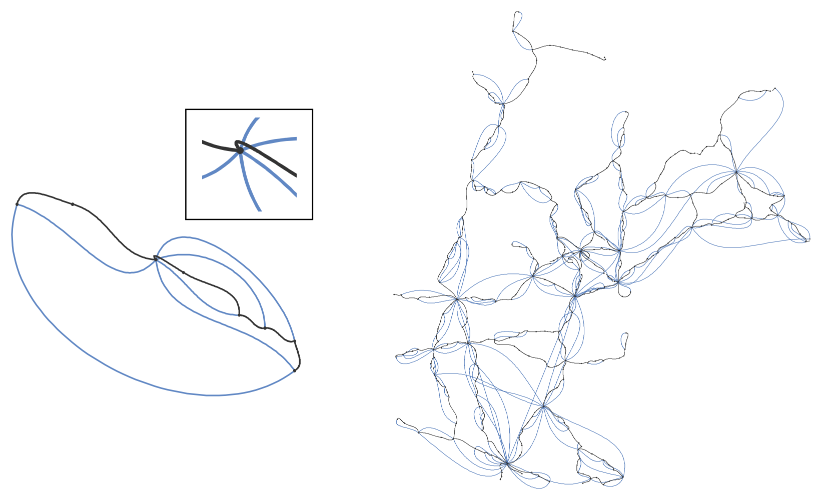
s t o o t o s ts s tis i t o st i ts is t v t it ot tio s s to p s t "sp tti" - i s. o ot tio is t to i ot tio s s i pi t o s s . p t tio s s to q ti st i t ss. p ti t ot tio i i i i t sq o s $_{i}^{d.\ v\ -}$ $x_i)$ is iq o p t i ti d_G $v))$ si p i o t o s. t is s it s tis s $_{i}^{d.\ v\ -}$ x_i . i s o s t t opti o so tio t ti s is too st o it io . ot o t t s t o- t o s ts sp i it i . tt i s o t i s t o t o s ts t ti s v it α_i π o st i to π o t o i t) to $\frac{\pi}{d.\ v\ -}$ o t ot s. is s t o o st i ts is t . o t t is to].



ult o t pl g p g lgo t] b l ot t o t t out l t pl t o o t gl

ot tio pp o pp i to ti t p s is st s t.). o o io s i p o t o st i t- i i s t p s i i . so s o t t t s ts ot ti s tis to o o p s t pp i tio . spit opti so tio st i t ss t i s i isp s p t s -s p s pp tt .

o t ot tio pp o s i i t p io s s tio i s i s t t o t st i t- i p s t tio s t sti o t q it o t i t o - i t o t pp o . t is s tio p s t o t o s o t s st t s t ot tio pp o t t i o to ti t p s.



• •• • • • l ot t o t t l t pl t o t t o t gl . ot t u o tu t o g o t l t.

s o ti t p o ti ssi i to i i t si- ti . i i i s t pi o spo to t i s t s o s i it p o s st i t ot to t si o t i p t to is p si t i o s s ppo t o t t o . oi to sto i t ot tio pp o to o sp i pp i tio i s s. ost i po t t i t o p p o ssi st p t t t - i s oo o i o s o t o o j ti tio i s t o i o t -s p s. o s q so tio st o o st i t to opti i tio it io . s o t t t sti is iq opti so tio i o is o o o p t s si s i t ot tio pp o .

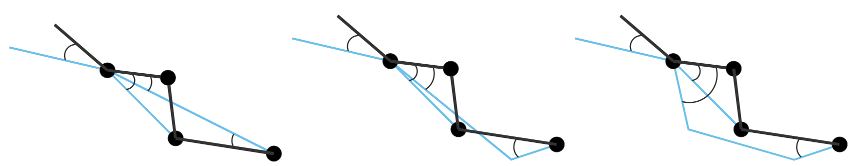
• •• • • • • • • • • • • •• •

i i i s p s t st i t i s t o i o i i s o t is . siti s i t t to s o t- t p t s o i i s. i tio s i t o s o t tt s t st i t- i o i o t siti s to so t it s s o st t i i .). o t ossi p o i st t i i . o i o t siti s is t i i t o st s i t st st o t o s ts it si i t t i tio s o p to t i t s o st t o t o s ts o o p p t i o .

• • • • • •• • • i iti o t o s ts i i t to t o p - o i to t i i t i st i t- i p s t tio is os st to o i to t si to i t p t i. . t t o t - o is o o t t i os st i i). si st t i i . 7)

i iti o t o s t o t siti is ssi to t t o i t si o its os st i i .

• ••••• • •••••••• • • ssi t is t to t o ossi s o t siti i i s. t t o i iti o t o s ts o si ot ssi to i o s t i t si o t i sp ti i i t ss ptio t t t s i i s i p t o i i s t i o t t t o - ist t i s i i t is t o) s sts t t t i o t o s t o t t siti i oss t is p t . t o ssi t o t o s t i i t to t t o s p s ss i po t t st tio) o i s o q s to t o t t i t s t o t st i t- i o ti t poi ts to ot o p s pi t i i . 7).



( ) g oup b lo t l g (b) g oup to o o g ( ) o t b l gt

• •• • • • t l o g o g . t t o l g o t ol g t g to

• • • •• • • • it i o p o t o s ts so t o i to t t o t st i t- i p s t tio o t i o spo i s t t t s o t st o s os st to t i ssi i i . i . 7). is o is i to oi ossi o j t t siti s.

s t i iti o t o s ts o t siti s i i t to o o t o p i to o o ps t t is t i t o i i s i i t to t t t . o p is ssi t t is t i i is to o t t t is its o is o o t o is i i i o sp o t . o i ti o t p p o ssi st p is ($_v$ $_V$ d_G V) o d_G v) E o E).

• •• • • • • • • • • •• • • • •• • •

it t is isti o i o to st t o o j ti tio o . t o i s t it i o o so tio st i t ss o ss s j t to st i t- i p s t tio o i i s. o si - i t st i t i i s to o t o s ts it i tio t c , . . . , c_d. $_v$ – t i tio s o o t o s ts i i t to t v i t o s ti o t p p o ssi .

opti i s a_i i $, \ldots, a_{d.\ v\ -}$
t o s ti o t o s ts i o st i t i t o-
t tio pp o) t i q i i i p t ssi to
t o p. tis tio o t so tio it io t p ss
i t s o t sq o it sp t to t t t s

$$A_v\ c) \quad \sum_{i}^{d.\ v\ -} \quad c_i \quad - c_i - a_i)\ , \qquad)$$

i i s o o $d_G\ v)$ p i s c_i, c_i o o t o s ts i i t
o ps o itt . t t o t o s ts o i i s i i t o
o ps.

o so s tio i t is ssio o t ot tio pp o
t i tio o o t o s ts o st i t s s o p i .
s t sq o s it sp t to st i t- i i tio s i. . t
o j ti tio o t ot tio pp o

$$S_v\ c) \quad \sum_{i}^{d.\ v\ -} x_i \quad \sum_{i}^{d.\ v\ -} \quad c_i - e_i)\ . \qquad)$$

o t it i ti t p o ts p o it o -
i t p t st s o t p i i s t o o t os
s p s t i s. s t o ss o i
t sq i i i tio o t t o i iti o t o s ts o
t st i t- i s o ti t poi ts. ot t t o i si
t it s o i tio t s t ot s t it opposit
si .

s s t o t p p o ssi o i iti o t o s t o t siti
is ssi to o p sso i t it t i t si o i i
i t ot is ssi to o p sso i t it t t si o
i i . si t s s o i tio it i o p sso i t
it s t si s t si o t i i s t
o s ot t t ot t o it i i t si o opti i s
a_i is s s). o - o ss is t s s

$$R_v\ c) \quad \sum_{i}^{d.\ v\ -} \quad c_i - e_i) - \ c_i - e_i))\ , \qquad)$$

c_i is t i tio o t i iti o t o s t t t opposit o e_i
e_i is t s st i t- i i tio o t .
o j ti tio o o t o p p o ss ti t p s is
o s i t s o t o it i $U\ c)$ $_{v\ V}\ \omega_a\ \ A_v\ c) +$
$\omega_s\ \ S_v\ c) + \omega_r\ \ R_v\ c)$.

t s o t t t o t o j ti tio $U(c)$ i t p io s s tio is i sio o t o j ti tio o t t i o t o s iq i i q i t ss ptio s.

o si t o o i t s o tio o p p o ss ti t p . o st t i t p G $V,E)$ t t s t o o t t o i iti o t o s ts o o t t o i tio s o t siti . o ti s i G j t i o is st i t- i i tio t ot is t o spo i o t o s t e_i c_i o so t) i t o s ti o t o s ts i so o p t t o ps o i t p p o ssi) o i t i iti o t o s ts o t s t siti .

ss t t o e u,v E t i ts $\omega_e >$ t t i s θ_{uv} $-\theta_{vu}$. it is s to s t t o o j ti tio st t s

$$U(c) \quad \omega_e \quad c_v - c_u - \theta_{uv}) \ . \qquad e \quad u,v \quad E$$

i t ss ti p op ti s o t is tio t s s t os o t t i o t o e u,v E $c_v - c_u)$ t o o i p s tt 's sis]. o to c $c_v)_{v \ V}$ i i i i t is tio t p ti i ti s

$$\frac{\partial}{\partial c_v} U(c) \quad 2\omega_e \quad c_v - c_u - \theta_{uv}) \qquad u \ e \quad u,v \quad E$$

st q o o v V. i t o st t to o 2 t is s st o i q tio s o i to t o

$$D(G) - A(G)) \ c \quad L(G) \ c \quad b,$$

$D(G)$ is i o t i it i t s d_{vv} $u \ e \quad u,v \quad E \ \omega_e$ o t i o $A(G)$ is t i t j t i it t i s a_{uv} $\omega_{\ u,v}$ i u,v E a_{uv} ot is b is to it o st t t i s b_v $u \ e \quad u,v \quad E \ \omega_e\theta_{uv}$. s ti t i $L(G)$ is t p i o t p .

$L(G)$ G

$$\omega_e \qquad T \quad e \ E \ T$$

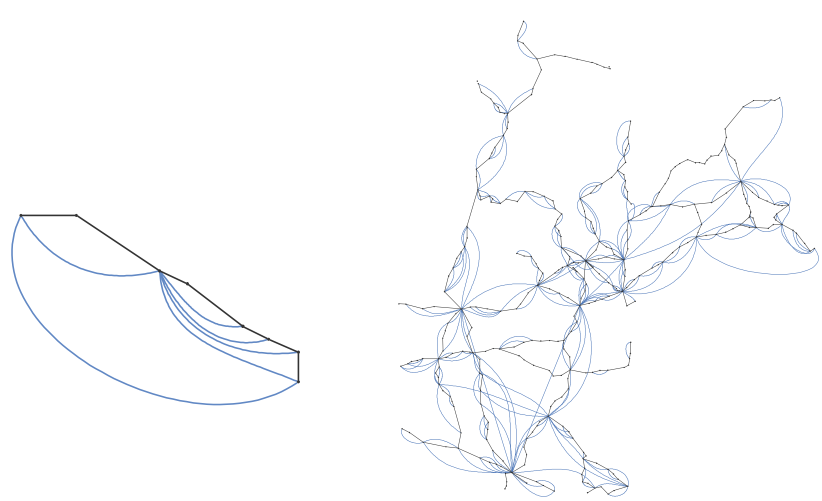
G $E \ T)$ T

ot t t i t i c o spo s to o itti its o o o L G) j sti b. i i t t i s o o t o t o G o - spo s to o t ti t s ti s t o itti t o o o t is t j sti b. o s q t t t i t o t s ti s t i is positi i t o t st o t i o t o - po t o G is . i itio o st tio is i i t o to t siti s o po t o G s t st o t t t o spo s to o t o s t o i i .

U c)

to t si o t pi s st s . .) ot o to so it t i ti sti pt o i t ti s st . i t t i L G) is i o o i t i st s ss- i it tio to q i pp o i t t opti i tio s. ot t t t is i o spo s to o - i sio sp i t t o s opti o t st p. iti i tio s t i q i i i t o t o i p i o i i is to o o p. t s o ts opti i t so tio it io s j t to t isti o i o pi o st i t.

i i s t s t o o pp o s pp i to t i p . p s i i t pp i s o t t o t o o t p o s t ot tio pp o i t s o is q it t o it sti o s ot q it t t q it o o - i t p ts.



out t 7 t t o (ω_a 2 ω_s ω_r)

ssit t o o o t o s ts it sp t to st i t - s is s t t t i o i o t o s ts so t t ω_s s o os si i t s t t ot t o i ts. i it t s o t t t st i t ss is s i t t o t p p o- ssi st p o it t is it io to t . ti oi o so tio s. o ss p s o p so p s. p s i t pp i s ω_a 2 ω_r. so o t t t i iti i tio s o i o so oi o i ts si t s st is t sp s t i ot tio o o t o s t is o . i s t 3 it tio s.

t pp o s o - i t o t ot tio pp o t p- p o o t is p p) i p t i ++ si]. i o p p oo -o - o pt i p t tio s o i -ti p i ts s o stoo s q it ti . i i tio is t ss q it . o ost o t ti is sp t o p p o ssi st p t t t i s t “ i o oo ” o o t o poi t] t o - i t pp o is s o i p o i so it sop isti t i p t tio . i t ot tio pp o is t st st o t i p t tio o t pp o p s t i t is p p p o s t i t ti sp t p o - s i s o tt q it .

u g t o u lt o k t t o (36 92 t). g p t t out p p o g

switzerland	22	32 3 (36)	3 ( )	.36	
italy	23 6	37 (9)	3 9 (2)	.	
france		7793 (2)	62 ( )	.	
germany	7 3	97 3 (9 6)	2 (3)	.	

s s o t o . it sp t to t p s t pp i- tio t o o o i t s o ost ti t t o ss i. . . i t o i s t p i o o t o poi ts t i i iti s ts i i t to t s t t o - i t pp o s si i o s ti o st t siti s to t s si o p t o i i s. s t to i t t t is t i t p s t pp o ?

i i p o s o t o o p i t o s os ti s p o o . o i o t - i sio i t p t tio i t o o] t t o t i to o t t s o o si s to tt t t s.

so i sti ti s st t i s o o t o s t t ssi - ts t t o s tis t i p op ti s o t s ti s i o t t pot ti so to p s t i t s i p it . is st ti o t t o o ti t p s t i i t i o o ti ti i i t i t o st t .

ot t t t u g t o p o bl t t t t t ’ g p to to t ult .

t i i t sti to is si i t sio s o t ot tio pp o o ot ppi tio s i i t t t fli t o t s o o - o p i t o s.

• ••••• ••••• •••• t i t o i p - ti t p p o ssi st p ti i o p i itio p i- ts.

• ••••••••

. . b llo . . . o t oot pol go l p t . ’ p g 3 pp. 62 99 .
2. . . k . . k . . lk . u l g t o k t . *t* *t* ( ) 6 2 99 .
3. . . *t* . o l o 972.
. . . ub . . p o g gul olut o u l t o o g og p t o k . *t* *t* ’ pp. 23 32. p g 2 .
. . . g . g g p l out to u l t t o t o t . *t* *t* 2 . o pp .
6. . . ook . . . t . . to . . utt . t o o t gl to qu . *t* *t* 7 3 2 3 9 .
7. . . g . . u . . oo . . obou o . g pl g p t ul . ’ p g 73 pp. 7 26 999.
. . . . oll k. o to pl g p o g . *t* 99 .
9. . . obk . . . out o o . . o t . pl t g g l pu po g out . ’ p g 3 3 pp. 262 27 997.
. . . . . o t . p o o t l out . ’ p g 7 pp. 36 373 99 .
. . . . . o t . . o. p og t t t g p . *t* *t* 7( ) 7 62 9 .
2. . . oo . . g . o k o g pl g p t u pol l . ’ p g 7 pp. 3 66 99 .
3. . ut g . ut l. l pol l g t goo gul olu t o . ’ p g 7 pp. 67 2 99 .
. . b . g . . t o og g bu l t t bl g p . ’ p g 66 pp. 32 337 999.
. . l o . ¨ . *t* *t* *t* *t* . b g t 999.
6. . u . o . l . . u l g t glob l topolog o t o . ’ *6* pp. 92 996.
7. . ul . g . . jk t ’ lgo t o l p l tu o publ l o t po t. ’ p g 66 pp. 23 99 .
. . . utt . o to g p . *t* *t* *t* *t* 3 7 3 76 963.
9. . . o g t b t t o o t po o t u t o . l t o ’ pp. 99 .
`http://rtm.science.unitn.it/alex98/proceedings.html`.

. t llo . ili . . olli

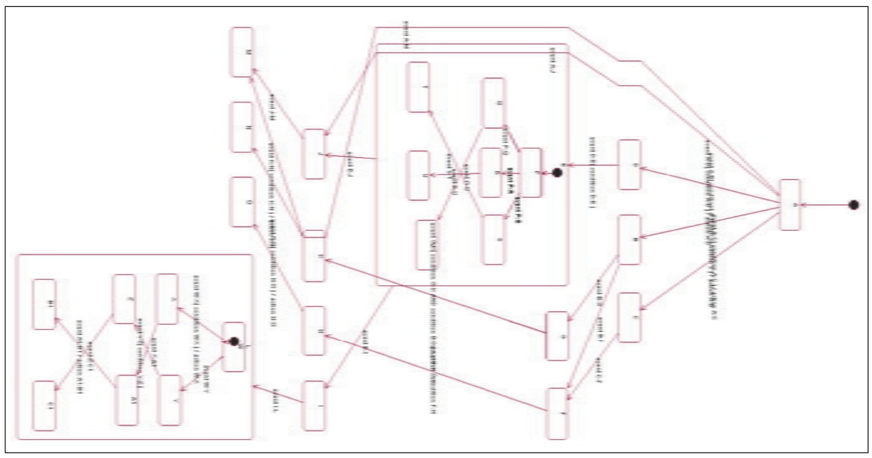
p rtm t o omput r i
iv rsit o s t ll s
o 3 6 i r so 7 3- 6
m il {castello, rmili, tollis}@utdallas.edu

t t rts 9 r i l us or t r quir m ts sp i- tio o r tiv s st ms. t is p p r pr s t r m ork or t utom ti g r tio o l outs o st t rt i gr ms. ur r m ork is b s o s v r l t iqu s t t i lu i r r i l r i g l b li g floorpl i g sig to ork i oop r tiv viro m t. - r or t r sulti g r i gs jo s v r l import t prop rti s t mp si t tur l i r r i l ompositio o st t s i to subst - t s; t v lo umb r o g rossi gs; t v goo sp t r tio; r quir sm ll r . v impl m t our r m ork obt i r i gs or s v r l st t rt mpl s. pr limi r r i gs r v r our gi g.

t t t [9 i p i l ot tio i l o t q i t p i- tio o ti t . o t i i i l p op t t t t p i i t o i li tio . i i t iti i o t t t o l b i l bl i to o t i o o l li to t o- t o t i i i ll . t i p p t t p obl o i li- i t t t p t l o it i o o p o i l i t iti i .

l i li tio tool o ti t p i tio i il bl i t t [9 26 . t o t tool lp l i o - i i i t o t t o tl op i ti t p i l ito t o l i q t o t o li o o pl ti t . o pl t tio l o tool [2 p o i t to l o t [3 t t t i . i o pl o t t t t t - tio l o l o t t i ppli . oti t t t itio l b l o l p t itio o l p it t t bo t i l b o - b o i . i i tio o i o t i i o l o it i o .

* s r support i p rt b i tio l bs b t s v s r rogr m u r gr t umb r 974 - 4 .



•••••• mpl o r i g o t t rt g r t b tio l os (r i g rot t b 9 gr s u to sp limit tio s).

o p i pp o to i i l i o i t p i - ib i i t l. [2 . l t io i tio o t i pp o b i t o i t lit t . o p i i i i [. t t io t t t i to o i tio l o o l (i. . t t p o t t o l l) i t o b o t l. [22 . t l. [7 p o i t iq to i t p i i pl -b l o it t t i ti to l t t ti t p o i t io to t b i l o it o i t l. b i - b . i i - o q pp o i ib b i t l. [7 to i p o t l o t-ti p o o l p o i ti o - l o ti . o tl o bi tio o t l o it o [2 it i t l-o t o o l i t iq p opo b [23 to to ti ll t l o t o l i . [l i i l o it o i l i p -li t t . p obl o i l t p it o t o i t i i [6. o t o t o t ( ) b o o l- b li p it o t o p i l t i l p [. o li olli [3 p t l o it o t p obl t t b ppli to i i l it o t . t l. [7 i pl pp o to ol t p obl o i i l i : t i l b l to t i l po itio o li . o t t t l- b l ll o ot o i t po ibilit o o l p it ot i o po t .

t i p p p t o o t to ti tio o l - o t o t t t i . o i b o l t iq t t i l i i l i l b li floo pl i . l o it o i -

i l i i i t o t l o it b i t l. [2 t t i t ilo to t t t . i l b l i l i ibi t itio i t t t lop l b li t iq . io l l b-li t iq ib o p i o p i l t -i l p it o t [3. o o t p obl o p i it fl ibl o t . i ll i o to t i p o t p t tio o t t t t i ppl floo pl i t -iq i pi b t o o t i i i tio o l o t [2 6. o pp o t i i l l b li floo pl i t iq i to o i oop ti i o t. o t lti i o l p op ti : t p i t t l i i l o po itio o t t i to b t t t lo b o o i q i ll . i pl t o o obt i i o l t t t pl . p li i i o i .

• ••••••••••

t t t [9 t it t t i to ib o t ol -p t o ti t . p o i i to ib o i tio o po ti l plo io o t t b i t t -o po itio . t t t t ot tio t t i ot b bo l b l i t pp l t o . i t to ot t itio b t t -t . t itio l b l t o $E[C/A$ E i bool o bi tio o t l ti li C i bool o bi tio o o itio A i -tio t t i t t t itio i ti E o C i t .

i t t t t b to t t o t t it t t itio . t t b p t l o po i to b t t i t o t o t o t o po itio . o po itio fl t t i i l t t o t t i i p t b p l tio . o po itio fl t o o i p t t t i i p t b plitti bo it li .

o pp o t t t i t t p . o [•] i t p o po to t t o po to t itio b t t t . o i l t ollo i i o tio : it it i t i t t oo-i t o it o i i poi t poi t to it p t t li t o it il it o po itio t p (. . o) t li t o i o i t li t o o t oi li t o tt ib t ll it li .

l i t t o t t t i / t t l ll t t . ll t i t t . oot o o po itio t o po to t t t t l o po to to i t t . ob t i t t b o po t o t o o po itio . t i o t p p t t l t i o tio i t t o t t l iptio o q i t to i o po itio t .

• t r m i r o t is p p r ill us t or s t i t r g bl .

t i tio ib o t t t i l o it . l o it
p o ollo : t t o po itio t i t i o to t -
i t i io (o i i poi t) o o i i .
o v i l t i p o i ll . i p o p o
l b l t l t t i io o t t l . v i
o t i l o it o t t t i o il o v
pl t i t to ot . v i o t i
l o it o t t t i o v il t i il to
p i l . o t o i pli it t o i o i o t ll
o l t to i t. i il pp o b to t ti l i .
o fl t o o i p t t t i . il -
o o t t l . i t o
o i q l to t i i t o it il t l it i t i
q l to t o t i t o it il t l . i l o it i
i pl t ot i t i t o . t i o o
p o t i i o t b t t o t t t i it
t i p l o o t i o t l b l it b o l
t t i o o l b o o ll . o - i t
i b obt i b ppl i t iq i il to floo pl i
i l o t [2 27 . ill i it t i topi i tio .
o fl t t o po itio o t t i to b t t . b-
t t o o t l . i (t
i io o t lo i t l) o o i obt i b i-
l p o i i i l i l o it [2 o t o o
it b t t . l o it t t o t t t i o o
t ollo i t i ti : (i) b t t i l (ii) b t t
i to l b i o i io o i l o it [2
(p o).
o (i 2) o i t o t o
t p :

. o t t i o b t t b t ti b t t poi t
b lli p o i p o ollo :

) i t o t t o po to t i iti l t t to t t l .
b) ppl - to i ti t o t t o p -
l t t po il o t .
) t l o i o v to p i l
i i t i b t l t o lo t p t o t t t o
to v. t t i t o i i x oo i t .
) ti to l it o i iti l l t t
ot i t l .
) i ll ppl o o i p o o p po i to i-
i i o i it i l . i o i p o i t y
oo i t o o .

```
( bj t ist o. il r )

  i r r   r  i g(o.  il r  );
  i r r  .  ig t   ;
  i r r  . i t    ;
  i     to  ( i r r   r  i g o  o.  il r  )
  _

  . l   r i .l rg st  i t   l rg st  i t  mo g t  obj  ts i  l   r i ;
  2.  (l   r i+   ≤  ( i r r   r  i g o  o.  il r  ))
     l  r i .l rg st  i t  s   o s t to t  origi _ o  v r  obj  t i
     l  r i+  ;
  3. l  r i .  ig t  summ tio  o   obj  t's  ig t  t l  r i ;
  4.  ( i r r  .  ig t   l  r i .  ig t)   i r r  .  ig t
     l  r i .  ig t;
  .  i r r  . i t    i r r  . i t  + l  r i .l rg st  i t ;
  6.  r  s  t  origi _ o   obj  t i  l  r i i  or  r to  l  it
     t   ig t o   obj  t   voi  ov rl ppi g;

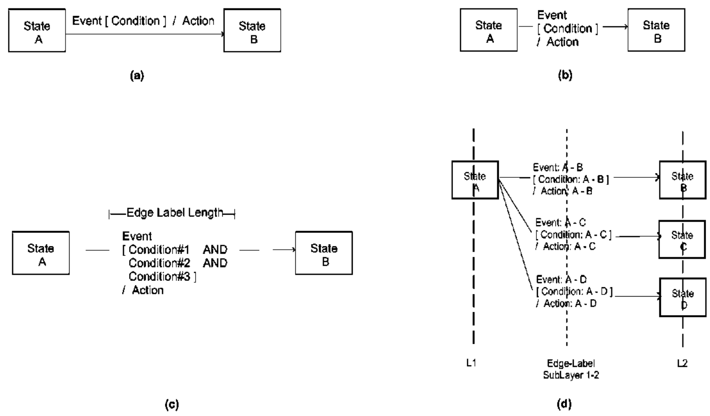
  _ ;
```

ro ur t t g r t s t l i r r o o .

2. i o po t i to t i t i io (i. . i t i t) o o i t i ib i i 2. lti i i to t i t i t i t o t p t ob t/ t t ll t oo i t o t o i i o t ob t t l .

o t o t t p o t l o it li ti - o pl it it p t to t b o o t p . l t t p o p o tt pt to b ti t obt i i b i t b o o i . pp o i b o t l p i [2 . ti - o pl it o t i t p o t l o it p o t b o ti t t i t o l . l L o t i $|L|$ o t t ti q i t l o it i $(|L|^{\bullet})$. l l t tot l ti o t i t p p po t i t ib tio o o i to l . t p o t bo o b pl b l o it t t i lt t t pt bl o t t t p. to p li it tio ot p o i o t il i t i p p . o o t il pl [.

t l b li lit t it i o o to i ti i b t - ( ) ( ). t t t t [9 ot tio p p i il o t o t p . t l b l pl t o o i t t t i t i pl : i o i t t l b l i ill t -



g l b l pl m t i st t rts ( ) l b l o si gl li (b) o l b l ompo t p r li ( ) l b l it l gt ( ) g l b l pl m t.

i t o i . o i *AND* o *OR* t t l b l i pl i t top l t o o t lo i t l .
o i o ol tio to t p obl o t t t . to- p t pl t o l b l t ti t ollo i it i [2 2 3 :

. l b l ot o l p it ot p i l o po t pt it it o i t .
2. pl t o l b l to t t it i i ti it t o i t i . o it t b lo to it o i t .
3. l b l t b pl i t b t po ibl po itio o ll pt bl po itio .

t t t t ot tio l b l o i t o t o po t : (i 3( )). o to ti t l b li it i i bo t ollo i t p :

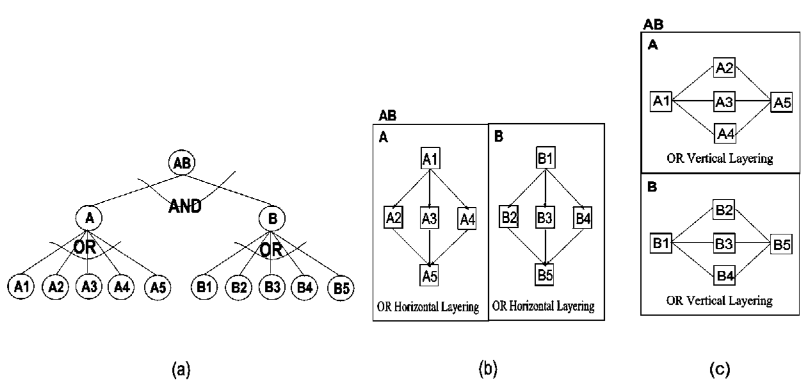
. t i l t o t l b l to o t t it t t itio t o po t (i. .) o t p t li (i 3(b)). t i o o po t i t t t i l t o t l b l t it it o l li (i 3( )).
2. t t b i i o t tio o t i l o it (tio 3) i l b l to bl (i 3( )).
3. t t i o l t to i t o i i t o t l *L.* *L.* t ti (i 3( )). o t *a* i *L.* i ti t t o *E.* b t *a* t ti i *L.*. o *E.* i t t pot ti l l b l o i o .

ti o pl it o t i t p i li it p t to t b o i t p .

• • •• • • • •• • • • •• • • • • •••••

o t p t tio o t t t it i po ibl t t t i *AND* o o t o po itio t l i o i io o t ot . ll t t o l o it pl ll t b o (ti ll) t to ot . i t o t lti i o *AND* o i q l to t i o t i t o t b o t i t i q l to t o t i t o t b o . i i pli t t b o bi tio o t o b o t l (o it l i t o it l i t) ill lt i i o t *AND* o t t o pi l . i i l l i bl . i- tio ll t p t tio o t i ot i po t t t ti it io i ot o t oll bl . t l t i p obl b ppl i t iq i il to t o o t i i i tio o ip [2 27 l fl - . loo pl i p titio floo t l i to fl i li t ll . loo pl o bi i t t t lo i t l o i i . floo pl i t floo pl i to i t l o t i t li t t i i t t l i to t o. floo pl i p obl i t ol tio t floo pl i li i [2 6 .

lt o o o l ppl t li i floo pl i t iq o i t *AND* o [2 to t p i l p t tio o t t t i pli t i t iq . ppl t li i floo pl i o pt to i t o i ti t t b ppli to t t t . o t i t t ollo i i it i o t t t :



• •• • • • - ombi tio ( ) / ompositio tr (b) v rti l sli- i g it ori o t l l ri g ( ) ori o t l sli i g it v rti l l ri g.

• to p t to i t t o i p ol l o t i l b l . i l b l ll itt o i o t ll (o bilit p po) ill l o i o t ll .

- o po itio fl t o i p t b plitti -t t bo i to b o o t b t t . i o o t o t i po t t t ti it i i p i i [2 oo to li -t t bo it o i o t ll o ti ll .
- t t b i i i l io i it o i o t l o ti l l i p i o t li i t p o t p t o (i. . o i o t l / ti l li i).

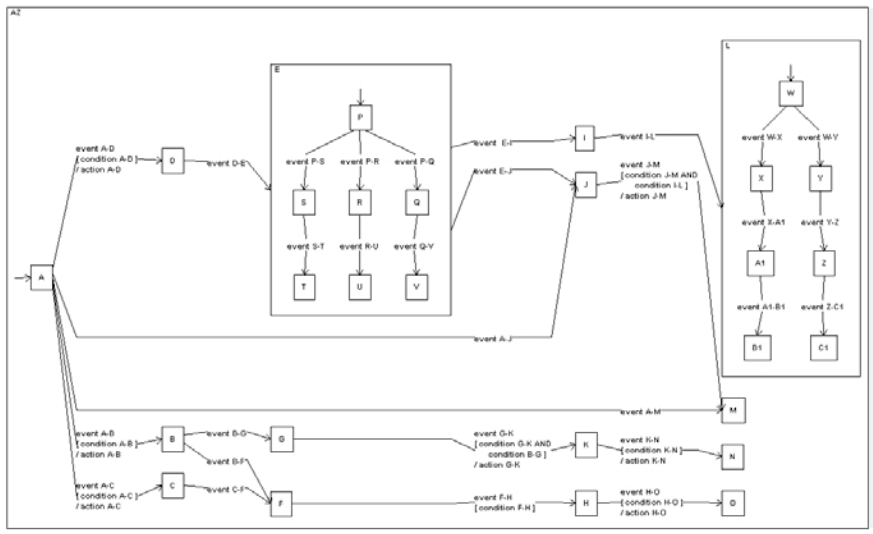
o l i to t i t t t o i o t l ti l i - io i i o i o to opti i t i . o t i t l i ti : / i ti ppli to t t p t i o t il o (i ( )). t o :

. p t o ( ) i li ti ll . t il o ( ) o o i o t l l (i (b)). t i t i t o t p t ob t i t i t o t i t il o t tot l i t o t p t i t o t il i t .
2. p t o ( ) i li o i o t ll . t il o ( ) o ti l l (i ( )). t i t i t o t p t o i t o il i t t i t o t p t o i t i t o t i t il .

i ti t t l t ot (/ / /) i il l o itt to p li it tio . o o t il pl [.
i o t t t t i t ppli o i p o t i t iq (i. . - o i -b tio l b li floo pl i) to t i o i . ob t t bot t o i- o t l ti l i io o i i o o ot o l p it ot i o po t o i b o t b o b b o i bl .

• • • • •• ••• • • • • • • • • • •• • • • • • • • • ••

t i p p p t l o it i o o t to ti - tio o l o t o t t t i . o i b o i i l i l b li floo pl i t iq . l l l o it o t p b pl it i p o l o it t lti i i p o- tool. i pl t tool i t l o it ib i t i p p t tool o o t t t pl . t i i t t b i io (it o t t opti i l o it) t t opti i - io . to p li it tio t i p o b o tool il bl t ttp:// . t ll . /∼ ili/ 2 /. lt ib i - bl . oti t t t t ppli tio o t opti i l o it ( )



m st t rt s i igur g r t b our r i g lgorit m it optimi tio t iqu s.

- o i o pl t l li i t (2) t b o -b i o i- bl (3) t i o oo p t tio. i opti i tio i p o o i bl t bilit o t i . o it o tit t i l bl tool to t p i o ill i t i / o o o i i t t l o p i l t t o t q i t to it l i .

omp riso o our mpl s o st t rts r b our lgorit ms

[illegible]	[illegible]		[illegible]		[illegible]		[illegible]	
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
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[illegible]

. . i ttist . s . m ssi . . ollis. lgorit ms or r i g gr p s ot t bibliogr p . *t* (4) 23 2 2 994.

2. . i ttist . s . m ssi . . ollis. *t s t s t s.* r ti ll 999.

3. . oo . obso . umb ug . *s* . iso - sl 99 .
4. . st llo . ili . . ollis. utom ti l out o st t rts. i l port - 4- iv rsit o s t ll s 2 .
. . s . g. r i g lust r gr p s o ort ogo l gri . . i ttist itor *s* p g s 46 7. pri g r- rl g 997. tur ot s i omput r i 3 3.
6. . s . g . i . tr ig t-li r i g lgorit ms or i r r i l gr p s lust r gr p s. . ort itor *s* *6* p g s 3 2 . pri g r- rl g 997. tur ot s i omput r i 9 .
7. . . . . outso os . . ort . o. t iqu or r i g ir t gr p s. *s t s t* 9(32) 2 4 23 r 993.
. . . s r . . ort . . o. g progr m t t r s ir t gr p s. *t t* ( ) 47 62 ov mb r 9 .
9. . r l. t t rts visu l orm lism or ompl s st ms. *t* (3) 23 274 u 9 7.
. . r l . ov r . m . u li . oliti . rm . tull- r uri g . r k t brot. t t m t orki g viro m t or t v - lopm t o ompl r tiv s st ms. *s t s t* 6(4) 4 3 4 4 99 .
. . r l . s i . lgorit m or blob i r r l out. *s t t s t s* '2 l rmo t l 99 .
2. . m o . ositio i g m s o m ps. *t* 2(2) 2 44 97 .
3. . . koulis . . ollis. lgorit m or l b li g g s o i r r i l r i gs. . i ttist itor *s* p g s 69 . pri g r- rl g 997. tur ot s i omput r i 3 3.
4. . . koulis . . ollis. t g l b l pl m t probl m. . ort itor *s 6* p g s 24 2 6. pri g r- rl g 997. tur ot s i omput r i 9 .
. . . u . tsuki. t v s i l out. *s t* 7 (2) 237 263 99 .
6. . g u r. *t t s t t t t.* o il o s 99 .
7. . . ssi g r . . o . . r . ivi - - o qu r lgorit m or t utom ti l out o l rg gr p s. *s t s st s t s* 2 ( ) bru r 99 .
. . ' o l . l t . rgstr . utom ti o or mb s- st ms b s o orm l m t o s. v il bl rom l logi ov r t t r t. ttp // . l logi .s /solutio /t p p. sp. ss o pril 999.
9. . t rso . v r omi g t risis i r l-tim so t r v lopm t. v il bl rom bj tim ov r t t r t. ttp // . bj tim .o . /otl/t i l/ risis.p . ss o pril 999.
2 . . ur s . i st ti s t gr t st t o um u rst i g. . i ttist itor *s* p g s 24 26 . pri g r- rl g 997. tur ot s i omput r i 3 3.
2 . tio l. os j v . o lo rom tio l ov r t t r t. ttp // .r tio l. om. ss o ov mb r 999.

22. . . o . vis . ssi g r . r. bo s r or ir t gr p s. *t* *t* 7( ) 6 76 u r 9 7.
23. . m . t i g t sugi m lgorit m or r i g l ss i gr ms o r s utom ti l out o obj t-ori t so t r i gr ms. . i ttist itor *s* p g s 4 424. pri g r- rl g 997. tur ot s i omput r i 3 3.
24. . to km r. ptim l ori t tio s o lls i sli i g floorpl sig s. *t* *t* (7) 9 9 3.
2 . . ugi m . g . o . t o s or visu l u rst i g o i r- r i l s st m stru tur s. *s* *t* *s* *st* *s* *t* *s* (2) 9 2 bru r 9 .
26. rtis o t r ools. l-tim stu io r tio l lt r tiv . v il bl rom rtis o t r ools ov r t t r t. ttp // . rtis s . om/rt i logu /p s/r tio l.p . ss o pril 999.
27. . im r . or . rb um. loorpl s pl r gr p s l out. *s* *t* *s* *ts* *st* *s* p g s 267 27 9 .
2 . . o li. logi o utom t m p l tt ri g. *t* 9(2) 99 972.

HERMES*

{carmigna,gdb,didimo,matera,pizzonia}@dia.uniroma3.it

HERMES

2\. 22] 23]

3\.

* HERMES http://www.dia.uniroma3.it/∼hermes.

t g l k "t o g o t t o t t o t t g t t o ot ".

o to g o t o t o t opt t o k p oto-ol ll • • • (o t oto ol) 3]. p oto ol o t t t t o o t t t lo g to t t t g o t o o t t o t t k o . g o o t g o t o t t to •••••• ••••••; o pl " o ot t to t p k t go g to t ot ".

l tool lop o l g l g t t t topolog t t l l 9 2 23 3]. o o op o tool t ll ot o pl t l t to ot t t t o t t t t o t g . o o t t go l o o g l g po t o o t t t t p t p o lt to (.g. 9]). t tool po t t tt t o o p fi o l o g t t t t o t o (.g. 23]).

t o k t ll HERMES llo to g t l t p o o t o o p fi to plo l t t o t o . t o HERMES t ollo g.

HERMES t t t t . t t t top-t l t oll t t q t o t to l -t . t l t t q t to q to po to (t otto t). t t top-t t plo l t t o t o l o t o o t t o t g pol . t t t g p (ll • ••) o t g p o ll t o t o . plo t o t p t p t o t o (t g).

HERMES l l g po to (o t). po to p t o -l o pl l t o o 2] . t t po to o HERMES to o t t t t o t o g p .

l -t o HERMES p l t g p g o l t t o p t t g o t t o t o l plo t p fi t . o l o t oolk t l 2] t ollo g t

t g o t o t po 2] o l o o t ogo l - g g t o g g t t o . o t l g p o t t ( ) o g o o t t t t t po o l t t o p - t g t .

t q pp t t o t g p g lgo t . t t plo t o t p t p t to . p - g o t t t o t t (o t) g t t to t t o lgo t . o t t t t lgo t t g k . lgo t llo to p t t l p o t 9 27] t l t t o

plo t o t p to g t t l l t t o o t t t t t lgo t .

t t lgo t o t topolog - p - t pp o] plo t t o p t o t q t t t t p 3]. topolog - p - t pp o o to t o l t p t l ppl t o].

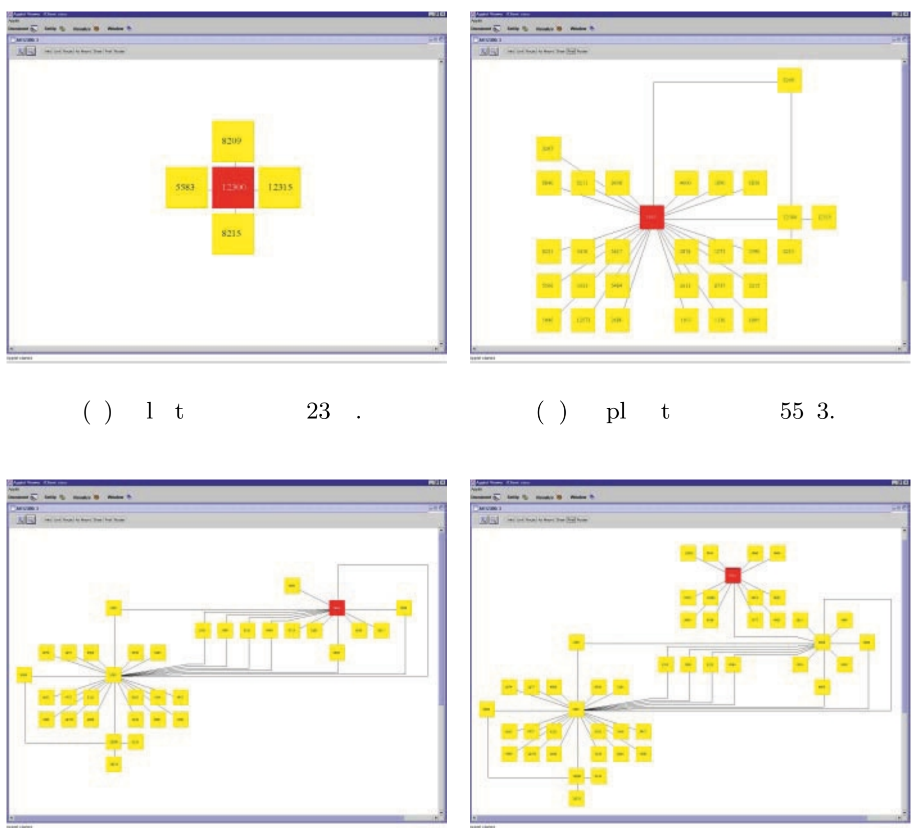
lgo t g p g lgo t t t o g to t o t t . ( ) t to o pl - t l t g t t t topolog - p - t pp o to po l to lt t t g t t g o t t t lgo t . (2) t to o t t t t t o o t po o l HERMES. (3) t to llo t o t . l lgo t tl p opo t l t t o g p g lgo t .] l t lgo t o o t ogo l g p - t t po t o o t t ot g t t t l pl t. 29] o t o o t t o t ogo l g p g t . o fi t g llo t o - o p t o t o o t g . 7] t t t o o GIOTTO 33]; t llo to t ll t g to o t ogo l g o t t t p o t o o p t o t o o - t g p t o t o t t.] t p t lgo t o o t ogo l g t t llo to p t l t po t o t o . t o g t t o o t g . t lgo t o o t t g g o g p t ll l p g t t l p o t o pl 26 2]. l o 6] t p t t o t t t t to l t t g t g t t o. o k o o o t t lgo t o ll t o t t ( ) (3).

p p o g ollo . t o 2 pl o t t - t t HERMES p o g l l pt o o t t o l t o t t . t o 3 g o t l o t t t t t t o HERMES. t o o t lt o t o t t o to g p p o o oo g t g p g lgo t to p- pl HERMES. p t l g o t •••• •••• •••• o t g g o t t . t o t g o t o t lgo t .

• • ••• • HERMES

t t t HERMES t o g • ••• •••••••. g p (l o ll • ••) g p o t g p o ll t t o - t o . p t ll o t t g t o po l t t g p t . o . o t p o t o pl ll

t t t o t to t. g. ( ). t o o t l t . t o po l t ( ) l t o o ll t o t t g p fi (g. 2); (2) l t o o ll t o t t g p fi p o . o t p o t o ll t o t o t t l t o t . l o t opt o o tt g t p p g ll t t t t (o) t l t t o t p op g t o (g . 2( ) 2( )).



( ) l t 23 .

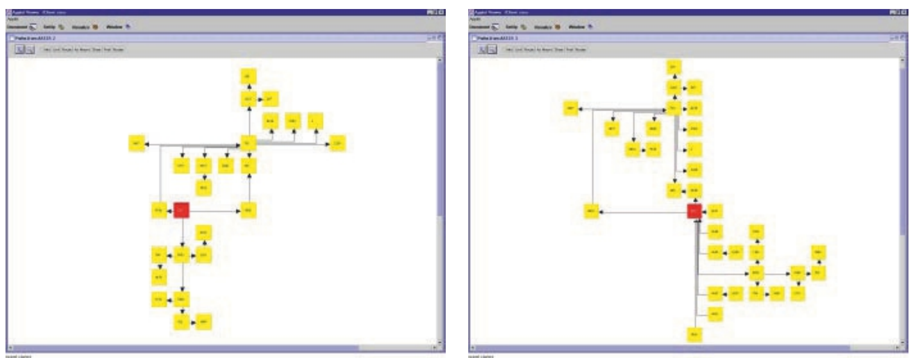
( ) pl t 55 3.

( ) pl t 5 .

( ) pl t 67 5.

• •• • • • pl t t p t p . l t l .

plo t p g t ollo g p t . u o g t o pl t t p l t . t p g t t ll t t t o t to u. t o v o t to u g (u, v) . t t o g

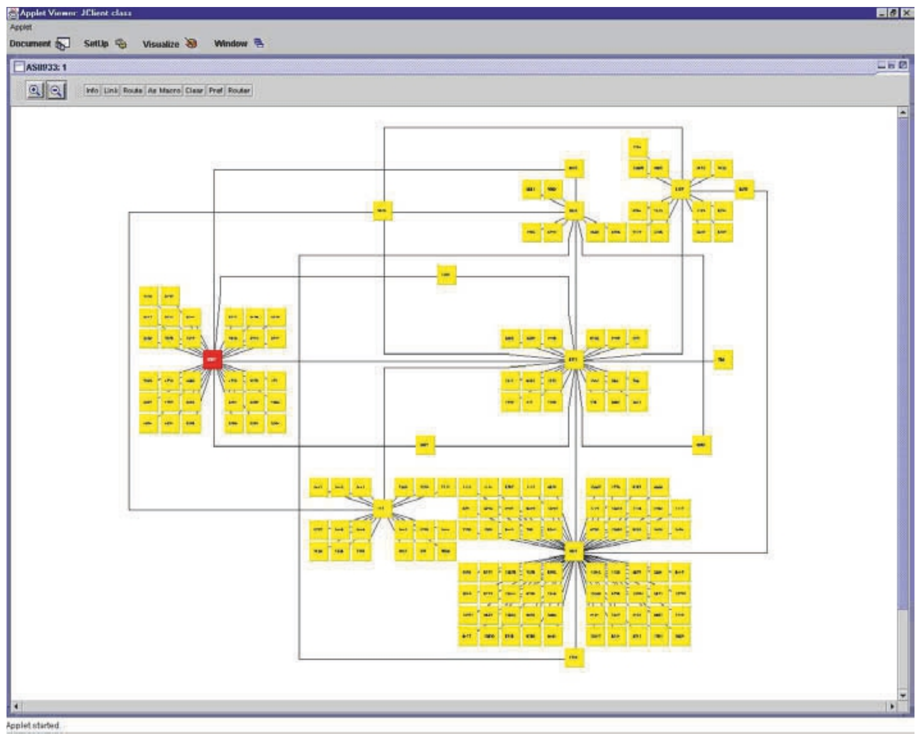


( ) p t l t - t 37. ( ) p t ll t t .

l t t ut t 37.

to t fi t o p g p o t t o t o g p t t ot g p .
g. o q o plo t o p t ppl to t p o g. ( ). 3 67 plo g . ( ) ( ) ( ) p t l . g. g l g t l t o HERMES. HERMES o t t g t g t t o g p g lgo t . - o t g o g. ( ) o t t t lgo- t t t g o t g o g. ( ). g o g. ( ) o t t t lgo t t t g o t g o g. ( ). o l t g o g. ( ) o t t t t lgo t . o o t lgo t to ppl o t t (t o) o o t . t g l g (t o) t p o t o t g lgo t o HERMES p l tt t o ot to t p t t o o t o g g . g. o o t t o g o pl o t t t . l o l t o t t o l o pl g t o g to t pl - po o l 3] (g. 2). o o pl p o t t HERMES p t g. 3. t o t o t . o k g o p p tl o t t o t t g t l o t o o

t o t o pt o . g. . o o t p o g t pol t o t. g. . po l ot o o o t pol . lt l o pl .
t o t k o o o t t t - o t t . g o t ot o t pl .

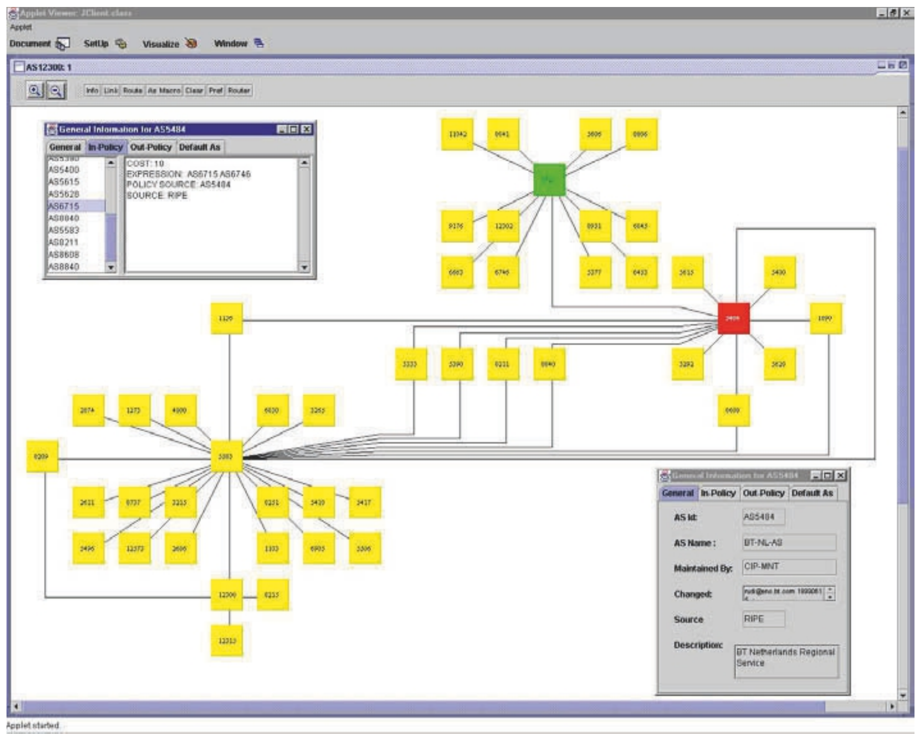


• ••••• p t t l pl t t p .

t o t o t o g t t . t l o po l to l t p op g t o o g o t t o po g t p. t o t o 2] l g t .

• • • •••• • •••• • ••• ••••••••

t t o HERMES t t . t t t top-t l t g o oll t g q t o g lt . q t o t l t to l -t g to p o t t t t o po to (otto t). l t lt - o t - ppl t o . t llo t to -o lt pl plo t o o t g p t t t . t olog to goo po t l t . p ot o t o t o 2. po to p t o -l o pl l t o o . t t o t t ollo g t opt g o p t g t t - l g g 2]



l ut p l 5 .

. t t o t g t p o
t o t p o t o t go t 2]. o t po to
l t l to ot t o .

t filt o t t o l t o t o HERMES to
t t t o o t k o g t o l p o
t t g . o ll o t po to o t . opt
t olog tl mysql.

l p t o t t l lo t t l -t . top
t q t t o t p o to t l t .
top t q o t o t p t p op t .
t top t q o plo t o g t k p.

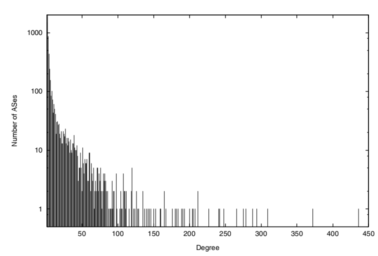
o q t p t ot p -
tl l t l -t . t o t topolog q t
l p t o . l t op o o o g
o . g o o t t p t t
o plo t o q t .

o q t tl p t to . to
o l g to t t t o t po to to o
g t o -t -fl .
opolog q t l t k l o t t . t g t
o t o o t to t g t to t p.
g o p t t o l (t o).
g g p l t oolk t 2].

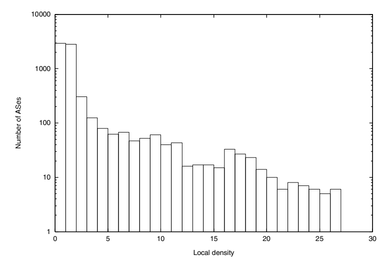
o to t g p g l t o HERMES l
t t t o to o g t q l g g p
G. t t o po l o t ollo g t t o G.
o t o G 6, 9 l t o g 27,6 6.
g. ( ) ll t t t t t o o t g o t t . fig o
t t l t t (o t 7 %) t g l o q l t
t l o l t o g o t . o po g
t l t o t t o tt t o t t g 62
, p t l . t o t t G o t 73 ol t t .
t o G . . o t “lo l” t g t .
o to t t lo l t o p t o t v
t t o t g p t t t to v. ll
g p . g. ( ) ll t t t t t o o t t o t
lo l g p . o t fig t po l to o t t o t % o t lo l
g p t g t t .
l o t to t t t po l t o t t plo G to
o t po t o o G t t lo ll . g. ( ) o o l d
o t t t p t g o t t t t to t o
lo l g p t t l t d. ot t t o t 3 % o t t
t to t o lo l g p t t l t .
o g o t t t g p o t o po t l g
t o t o 73 ol t t . o t 6, 36 t ;
o t g 6 o po t l t 6 t .

t plo t o t p HERMES o p t g. l t
l t t p t v ll t t g o t to
v to t p p . p g o t
p fi t t o t o t g lgo t .

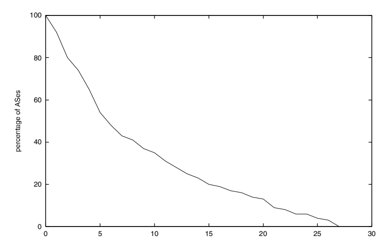
t p o pl t l t t -
t g .



( ) t ut



( ) t t ut



( ) t t
t l l p
t l t lu t

• ••••• l p t t t t p .

t g to t t
g t t t p o t t g g t po t o
o t t g t p “ po l ”.

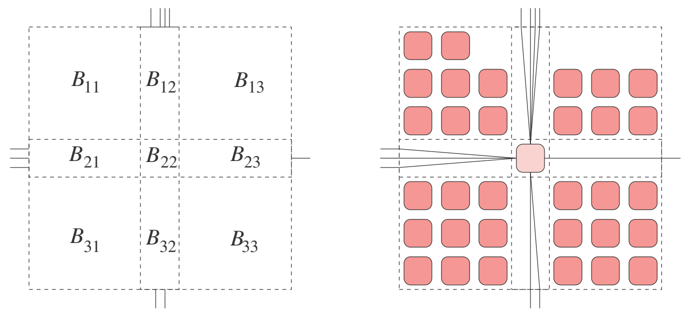
o t t t t lgo t t g -
k . lgo t llo to p t t l p o t t
l t t o plo t o t p to g t t l
l t t o o t t t t t t lgo t . t t
lgo t k o lo l opt t o t t g . opt t o t t g
o t t t lgo t glo l o t . o t t t t
lgo t t g q t t o t p o o t
t l p lo t.
o t o ot t o to t ll oo t t t t
lgo t t lgo t p g o t t k o
g t t t t plo . o t
l o t t to ppl o o t t o lgo t . l
ppo v t t t t to plo . o o t g

lgo t to ppl o l l t g •••••••••• •••• o v o p g
o t t t ol t t t- p o fig t o .
plo t o o t o p t ollo . o g t o t . o
t t t g t ol g p o g t
g p t o t . . t o t l o t o o t
o t t lgo t ok o . plo t o o t
o g t t o t t o t lgo t
t p t to t plo t o . o t t g
t t p l ll l t pt o o t lgo t .
t t lgo t o t o t ollo g t p .

t o g o t po l o-
.

t pl t o] t q ppl .
ppl t o o t t -
q p t 3] o o t t g o t ogo l g (t pl -
po o l) t t o p . o p t g
t v t gl . g t o v t o o l t
l po t o t . l gt o t o
to o g p to o o t ll t t t t
t po l o t fi t t p t t t to v.
o p t g t v p -
t t o to t gl g to t o t ol .
ot t $B_{i,j}$. t gl B , o g v. t gl B ,
B , B , B , o g t g -o t t to
v. t g p t t t g t-l g t po l
o l pp g ot g -o t . t ll t o t k
o t t . t gl B , B , B , B , o o t g t
o t o o v to t ot t . g. 6.

g lgo t llo to ppl t p t o t -
t p. g to t p t t t o
t u v; t u v t l t t p.
t v to t p o t to u t
g (u, v); t u t l t t p.
t to t p pl tt g g (u, v); g (u, v) t
l t t p.
lgo t o p t t po t o o t t g t g
to opt l t t (o o g o
g l gt) t t t p g o t o t t o
o o t . o t g g t to t p
o t t t g o o t o o p t .
t ollo g g o t l o t t t t t
o t t g p t o k. t o g o t po l
o to t t t t t t q o .
t op o t p t t ollo g t t

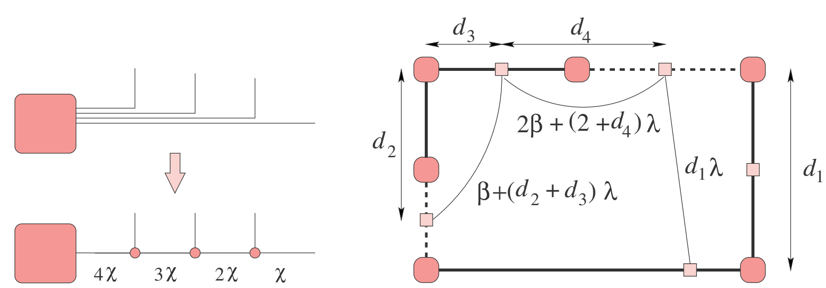


( ) ( )

t p t t . ( ) . ( ) t
t t - t .

ll t o g o t p pl t o
t t t topolog o t g o pl .
p pl fi o t t ll t g l o q l t o .
o t t t q opt t po o l 3]
ll t g t o t t o t
oll p to o gl g . g. 7( ). o t t
o t g p t g t o g t pl .
t o t o l ppl to ll t t .
g to t p o o po g (l g t
t l o) to t gl t t l t lgo t 32].
ll t g t g o t o g l p.
D o t t . t o k fi ol-
lo (g. 7( )) ( ) o o D t (ol) g
o t p. (2) o o po g to ol g o t o t
q l to t t k lt pl o t t χ (g. 7( )). o
o po g to g o t q l to o. t t l t o t
o t t g p t t o t o o g ol g t t g .
(3) to D o p o g o t p g .
g. 7( ) o t o D. ( ) t t o o o t l (t l)
g t t l o t o t o t t t q l to
t t l (o o t l) t lt pl o t t λ. g. 7( ).
t t l t o t o t o t t p p t lo
o o t l gt o po l g t t ollo t . ( )
t o o t l t l g o t t t q l to t o -
t ogo l t t t t o t g lt pl o t t
λ pl β. g. 7( ). t t l t o t o t t o t
t p p t lo o o t l gt o po l g t t
ollo t pl o t o . (6) t t o o o t l

(t l) g t t l o t o t o t t t q l to t t t t t o t g pl 2 (lt pl o t t λ) pl 2β. g. 7( ). t t l t o t o t t o t t p p t lo o o t l gt o po l g t t ollo t pl o t o t o . (7) t t l χ β λ p t t o t o o o o o t o l gt p t l . l t- p t . t o t χ β λ t t o o t lgo t .



( ) ll p t - t t t .

( ) (l ttl qu) t t k.

• •• • • • llu t t t l t .

t - g tt - t t- t pl t ollo .

t u v ot o t t o t po o to D p t g u v. l o t po to D t u t o p t g t t g . o o v. t po o o t. o t t p t t u v o p t . p t t t o t t p o t g . l t g t t p ollo g t o t o t t p t . t po o o t o g t t g o po to t gl . t u v o t pl t q (ot o t) opt . lo l l t o o t g t o u p o . g p t p l t o t o o u t o t g o o g . g (u, v) t pl t to t o p . t g p t t t p l o t t .

p t p o t p o t q
o ppl to k oo o t t o g o t t t po-
l o to t t t .
t t D o t t q t t
o g o t p. o t po l to t t D pl fi to
q l t t t l o .

g t l to oll o to o o-
o t o t t l o t t o t pl t t o o t
lgo t . l o g t l to tt o l o o t
po to .

. p . p t u . l . ttp // . p . .
2. . t . . . . u t . . p t
. u. p t t p ut p l ut t . l
. p - ttp // . p . t 7 6.
3. . t l . tt t . . put t l
t t u u . s t s t s ( ) 2 .
. . . l . u . - t t t t l p -
. . u k . t t s
lu 2 t t s t p 37–52. p - l 7.
5. . . . t .
.- . . . t lu 533 t t s
t p – . p - l .
6. . . . t t t t l p
l t . . . t t
lu 5 7 t t s t p 57–7 . p - l .
7. . . . t . . . t t -
tt l t t t t l p . . tt t
t lu 353 t t s t
p 3 3–3 . p - l .
. . uk . . t u t t up t t upp t
. 2.2. . l . p - ttp // . p . t.
. . tt l t p l pl . l . ttp // . . .
. . l kt u l l . l .
ttp // . . .
. . . . tt t . . . ll . p -
-p ll l p pl • • - p . t
2 (5) 7 – 5.
2. ll t . u . l .
ttp // . . ll. u/ /t p l _ / / u . t l.
3. . tt t . . t . . t l
qu -up t t p . . t l t
lu 73 t t s t p
2 7–3 . p - l .
. . tt t . . . . ll . . t
ll pp l .

5. . tt t . . tt . . . u.
p t l p u p l t . *t*
7 3 3–325 7.
6. . tt t . ll . t l . t l u t p .
. . t t lu 5 7 *t*
t s *t* p – 5. p - l .
7. . . tl p . l .
ttp // . p . / tl / tl . t l.
. . . . . . u . l t p
t . . tt t t lu 353
t *t s* *t* p 33 –335. p - l 7.
. . . . u . u . t t l p
. *s* *s* p 2 –33 .
2 . . ¨ . u . p t l u -
. . . u t lu 27
t *t s* *t* p 25 –266. p - l 6.
2 . lk t . p t lk t. l . ttp // . t lk t. .
22. . u k . l t u l t t t ult t k . l .
ttp // . . .
23. . t t p u t l p t. l .
ttp // . t. u/ p .
2 . t t k . t . l . ttp // . . t.
25. . . t ut p t. l .
ttp // . t .u . u/ ut - .
26. . l . . k . . t l pp t t t
p l ut. *t* *t* *s* *t* *t*
3.
27. . u . . . u . ut u t t t t l
p. *s* *t* 6(2) 3–2 5.
2 . . t . t l l ut . . . u t
lu 27 *t* *t s* *t* p –
. p - l 6.
2 . . p k t . . ll . t t t l p .
s *t* *s* *t* *s* 7( ) 2 7– 3 .
3 . . t. t pu k t p l . l .
ttp // . . ll. u/ /t p l _ /t p l / ult. t l.
3 . . k t . t p t l (p-). 77 .
32. . . p t t t u u
. *t* 6(3) 2 – 7.
33. . . tt t . t . ut t p -
l t . *s* *st* - ( ) 6 –7 .

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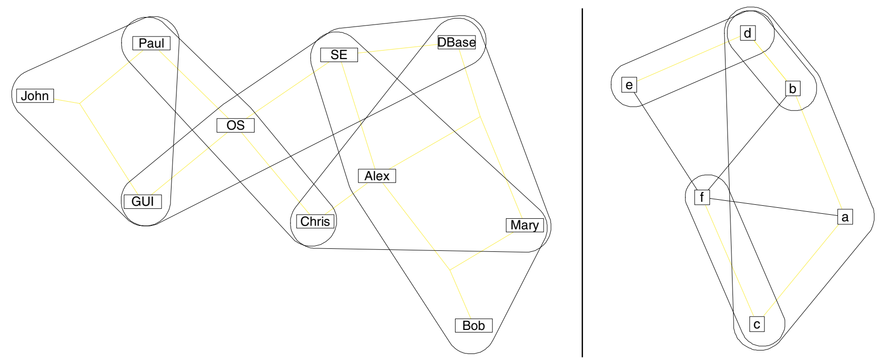
r p rt n p r nc n pr ct c l t r r n p r r p n t u t t n r . PATATE t n t ppl c t n cl c l rc r ct t t n c r p c uc t n t r t nt r t p r r p truc tur n p rt cul r rt l c t n . r nt tr t t n t n c un rl n r p r pr nt . llu tr t n p rt cul r t t n t r p t n c put n n ucl n t n r tr .

p r r p s c s t so o c ss c r p s c s c r pr s t r t o s ps t or t t o rt c s. r s p o poss r p c r pr s t t o o p r r p r c s o r t o s ps r r pr s t c r s t t c os o . t rpr t t o o t r co t t o r r t s r s t s r o t s o r r s k o o t s . o p p rs t t r t r st s t r o p r r p s 7 .

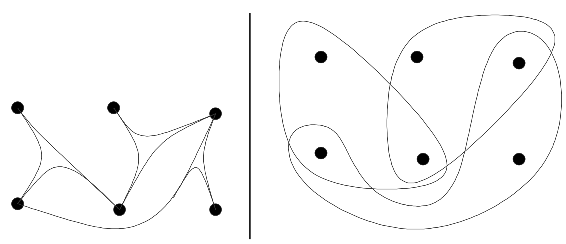
o so o k tro c t o ot o s o p r t o p r r p s sp r t r r pr s t t o s o s ts 9 co p t ss r s ts . t p r s r r pr s t c s o p r r p t t s t s so co ct t prop rt .

¨k tro c t o k s o p r r p r s 7 . ot c s s rt c s r r pr s t po ts t p . t p r *e* s r pr s t co ct t po ts t t r pr s t t rt c s t t *e* s oot c r s. o rt c s o to t s p r t r s s oot c r t t po ts t t r pr s t t s rt c s. t p r s r pr s t c os c r t t co t s c s t po ts t t r pr s t t rt c s t t t p r . r (t s p o r o p r r p s t s t o r pr s t t o s. t o or r p r r p s t st r s .

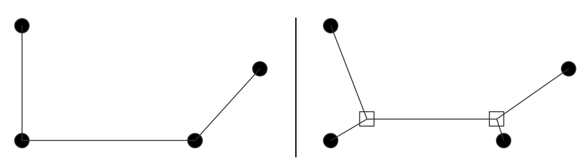
r n p r r p n t u t t n r



pl p r r p (l t) n r p (r t) r n t PATATE .



r n p r r p n t t n r (l t) n n t u t t n r (r t).



pl n u ucl n p nn n tr (l t) n ucl n t n r tr (r t).

r PATATE s st oc s s o t r pr s t t o o p r r p s t s s t st r . t s r pr s t t o t p ss s o t r pr s t t o o p r r p s s s t t rs ct o s. s r pr s t t o s so st p to r t r o r p s str ct r s tro c r 3 . r p o c s p r r p str ct r c s t o s or p r s c . PATATE c r r s or s s t o r p s t s c t o s o r t r pr s t str t s. r (r t s p o s c r p r o t t PATATE.

• • • • • • • • • •

••••• $G = (V, E$ s t s t V o •••••••• t s t E o ••••• t t s or r p rs o rt c s. •••••••••• $H = (V, E$ s t s t V o •••••••• t s t E o •••••••••• t t s or r o pt t s ts o rt c s. t s p p r co s r••• •• p r r p s t t s p r r p s t t co t p r s t t st t o ts. t o p t PATATE s s o o s

ss r o oc t o s to rt c s o H
or ro t r t o s
o str ct r p G ro t c rr t pos t o s o t o s o p r r p H s o o t t r t o s scr o . or r rt v H t r rt $\nu(v$.
t t oc t o s o t rt c s G to t oc t o s o t ssoc t rt c s H ccor to ν.
c pp orc r ct r or t to co p t oc t o s or t o s G.
t t oc t o s o t rt c s H to t oc t o s o t ssoc t rt c s G.

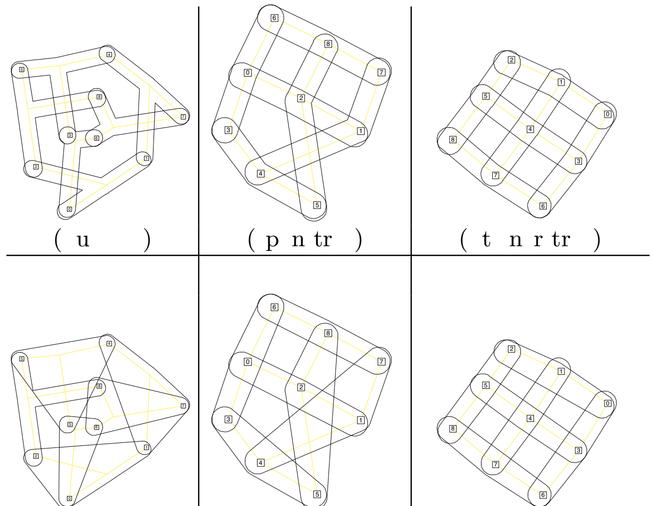
3 or c p r $e = \{v_\bullet, \ldots, v_\bullet\}$ H c r t t s o t s t co to r o t o o t s G t t ot s e t o o t o s s t o o t r o t s t ck ro s. pr ct c ppro t t t ck s po o s.
opt o st p (co s to co p t or c p r e t co $ch(e$ o t c r ssoc t t e. c r o e s s t to t co $ch(e$ o rt c s t t e r c $ch(e$.

ss t t t r r s r t c sp tr s c t r tr s (s . r 3 s p o s c tr s o p rt c r s t o po ts. p r r p $H = (V, E$ t r r p G t scr pt o o t PATATE t o o c s o o t t r o o t o s st rt t pt r p G

(or c rt v H rt $\nu(v$ s G or c p r e H rt $\nu(e$ s G.
or c p r $e = \{v_\bullet, \ldots, v_\bullet\}$ s $\{\nu(e, \nu(v_\bullet\}$, $\ldots, \{\nu(e, \nu(v_\bullet\}$ r G. oc t o o $\nu(e$ s s t to t r c t r o $v_\bullet, \ldots, v_\bullet$.

(sp tr or c rt v H rt $\nu(v$ s G. or c p r e H c sp tr t t co rs t rt c s t t e s co p t t s o t sp tr r to G.

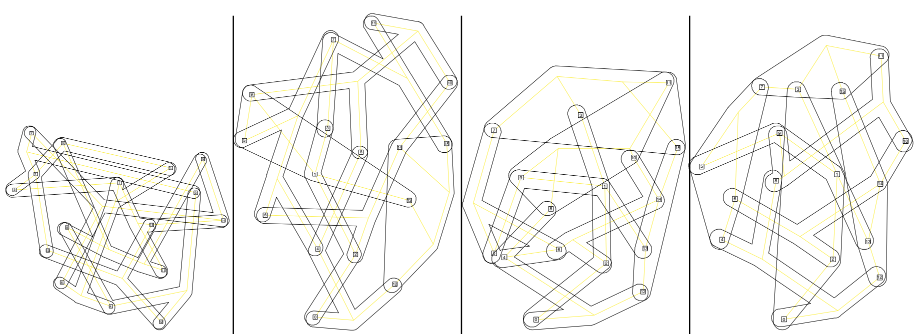
(t r tr or c rt v H rt $\nu(v$ s G. or c p r e H t r tr os s r t rt c s t t e s co p t . rt s to G or r st r po t t s o t tr r to G.



(u) (p n tr) (t n r tr)

(u c n) (p n tr c n) (t n r tr c n)

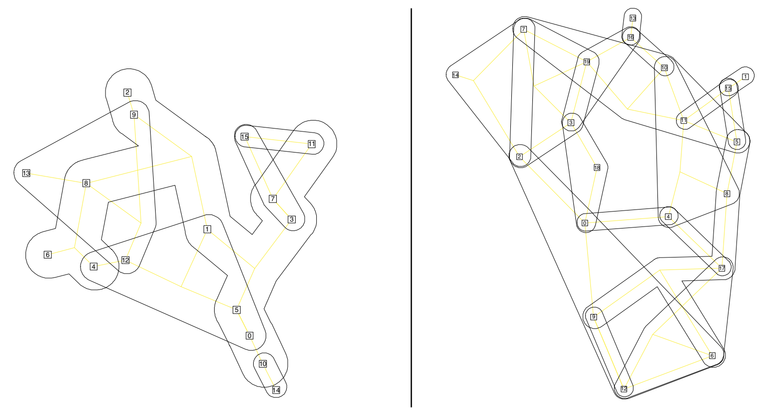
• ••• • • p r n t n t r r nt t (u p n tr t n r tr) t c put t un rl n r p ur n t t r t n t p r pr nt t r t ut t pt n l (c n) t p.



(t p) (t p) (t p 3) (t p)

(t p) (t p) (t p) (t p)

• ••• • • lut n t lut n t c t r t n t p t PATATE t n t t n r n c n pt n r u .



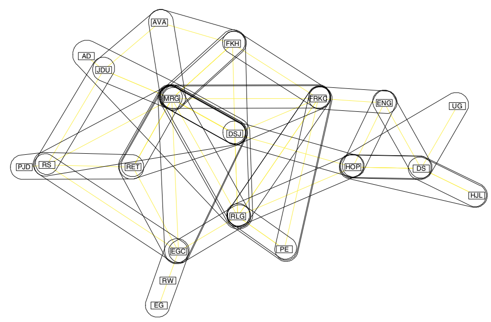
r n n p r r p t l n p r (l t) n n p r r p t n p r l n t t t n rt c r t n t t 3 p r (r t).

PATATE t o s p t ++ s s t r r or t t o o t r p str ct r s co p t t o o sp tr s co s. ost r s st s s to co p t t r tr s. o c t o o t orc r ct or t r c t r o s s to co p t t oc t o s o t r r p G. o tp t o t pro r s postscr pt t t r pr s ts t r o t p r r p s p t.

r r pr s ts r s o t s t s t r t o s t or t o t t opt o (co st p. r r pr s ts t o t o o t so t o t c t r t o st p o t t o t t r co opt o s r s .

t o sr so r s ts or s p r r p s t c s r rt o s to r o p r s (.7 . t r tr opt o c s co pro s t t (sp tr (opt o s s s to pro c t st r s ts pr ct c . or p r ts r ot st t t r or ot t s o t r tr s r c s t r o t r t o s r q r to o t r so r . t r t o s s c s t s p r t o o t o s ts o po ts c r s t p r t r co s st o t s p (co opt o .

r n p r r p n t u t t n r



r n t r ¨ p r r p r u n (p n tr) n (c n) p t n .

. . rt ult. rc r ct l r t t t pr r cr n pr p rt .
( ) 3 .

. . ruc t r n n . n l . r p r n rc r ct pl c nt.
( ) .

3. . r l. n u l r l . 3 ( ) 3 .

. . . n n n . . ll . p r r p pl n r t n t c pl t r n nn r . .

. . . rp. n t c put t n l c pl t c n t r l pr l . .

. . l tt . p nl u . ¨ n n n . r nt . pr r n pr ct r n t pr n l r t r r n p r r p . (3) 3 3 .

. . ¨ n n. t r p r r p . .

. . l rn n . r. l t r r n t r l n tr c put n . 3 ( ) .

. n nn. n t r t c n c n c l r pr nt t n pr p t n n r n n . .

. . nt r n . c r n. ucl n t n r n u tr n pr ct l r t . 3 .

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. l

ch l mput g th m t c l c c t ch
k ch . C.Walshaw@gre.ac.uk;
http://www.gre.ac.uk/~c.walshaw

c b h u t c m th g g ph h ch u mult l l t ch qu c mb th c ct pl c m t lg thm. h mult l l p c g up t c t m u th clu t t g ph p t u t l th g ph ll b l m th h l . h c t g ph th g t l l ut th l ut ucc l ll th g ph t t g th th c t g th th g l. th th mult l l lg thm b th cc l t g m gl b l qu l t t th c ct pl c m t. h lg thm c c mput b th 2 3 m l l ut m t t t umb mpl g g m t 22 t c . t l t c c mput 2 l ut p g ph u 3 c t g ph t u m ut th l g t g ph. h m g tu t th c t mpl m t t c ct pl c m t lg thm .

r p r b bl t olo t r t o t o t r-r l t t b pro ' l o t (o pr r b o 2]). r l l o t l or t r b o p l o l t rt r pl o to t ' r t p l t . p ll l or t r ll bl to pl tr t r tr ll r p b t r r t .

ot t o b o r ppro to t probl r ro o r or t l o r p p rt t o . r t r t b r o t t t o bot l r t r p p rt t o l or t p r p or -port tl t lob l p r p t to lt l l t q . to t p r o rt to or t l t r to r p r r l t r t t pro r t l t r p ll b lo o t r ol . o r t r p t p rt t o (po bl t r l-or t) t p rt t o l r o ll t r p t rt t t o r t t t or l. q o o tr t o ollo

b r p t p o /r t loop o lt l l p rt t o b ll lop tr t or o r o t lo l t r o t r - ot r opt t o l or t . .]. lt l - l pro l o r tl b ll ppl to t tr ll l probl pp r to or (or o b tor l opt t o probl t l t) b pl oot t obj t t o] t p rt or lob l p r p t to t opt t o .

t p p r ppl lt l l to or - r t pl t ( ) l- or t . t b pr o l t t r p r l t r t r or pl r t r ol 7] t t po bl o ' lt r t q t t llo ol port o o t r p to b o l t o r l] t lt l l ppro to ' p t t l t l] pro . or r tl r l] p r- t l r r l or 9] t ll lt l l (or t r r to t) o b t o t l or t r bl to ro- b tl l r p o 3 rt (lt o t r l or t t ll o t $O(N$) o po t). r ppro lt o r p tl (r t l or t) r t r t t l or t o tl r o r t t t lt l l p r b po r l tool or or r t pl t.

r l t b t o t r t t t o lt l l r . . 3]. t r t t lt l l pro to r t oo l o t o t or l r p lt l l r p r t t r b t r l l t r t t r p or b rt l l r to t o ppl r . l l r o pl t r t t t tr tr t r t b to r t o t r p t lt pl b tr t o l l].

lt l l l or t o tl r (ll r b]) or b r r l o r t r p t l t ll b lo o t r ol . o r t r p t t l l o t t l o t l r o ll t r p t rt t t o r t t t or l. l or t o ot t ll op r t lt o l o lt pl l l o t r p (or pl lt r l or t t) b t t r t l o t t l l t t rpol t t r lt o to t t l l o .

$G_l(V_l, E_l)$ t r r to r t o r r r pr t t o G_l (V_l , E_l) l t r l or t r - t r o r r t t l o r p r . . 3 2]. t l t r l or t to r t t or port t tr t r l t r o t r p or r t t t l t o o l l l t l. o r r r o l t r t t r o t or l r p .

t t (. . ot r l opt l) l or t t t
o r r ll (r l t r pr t t b t o t l-
t l l p r) or l (t o r o l ot t r t
prop rt o t r p r tl b t r t r o).
o t t r q r t o r ppro o
rt r t t t o t bo r o t t l t r r t
or o t o t 2 rt . o p t t q l t to
l p t b t o r p r t oll p to r t
t o r r r p . t p t o 2 t t r t
o t rt (o o 2 t t r j t) l o
or b to t t t o t br t p r t r o .
o t l t oll p t rt u , u
V_l t t r o t r r to or rt v V_l t t
v u + u .
probl o o p t t o t rt o t -
r l t t probl . lt o t r r opt l l or t to
ol t probl t r o t l t $O(N \cdot)$ . .]. ort t l t
too lo or o r p rpo t ot too port t or t lt l l
pro to ol t probl opt ll r t o t o tr t o
r t propo b r o l]. r t o o o tr t
t to r t r o l or r l t o t rt t t
t r t t rt t t bo r rt
(or t t l o t bo r t). t rt r r o
ro t l t. t r r r l t bo r t o o to
t t b r o b t or r to p t o r r p or
po bl t r o p r t to oo to t t t -
bo r rt t t ll t t (ot t t t or l r p
G t G_l or $l >$ ll b t).
o tr t t r o r p t l t b r
o rt t o r t r p ll r t o t r ol t or l
pr t o t lt l l p rt to tr t to rr o t t l p rt to .
t r o r p r t lo to o p t t t l l o t. o r
t r p o r o to 2 rt (b o t
o t o r ll b o t b l t) pl
pl t rt t r o t o lo o r l t . ot t t o tr t o
o to 2 rt o l l b po bl pro t r p o t
].
r t l o t o r p G_l t -
t rpol t o to t p r t G_{l-} . t rpol t o t l tr l tt r
t p r o rt v , v V_{l-} r pl t t po t o t
l t r v V_l r pr t t .

t l l or - r t pl t ( ) or pr - b r l o-
r t to r t r p G_l or port tl to pro t l po t o

or t p r t r p G_{l-} . or l o pt ro p p r b] b o t o r pl b pr . rt r t l po t o ll r o t t r l o t t t pr o t rt to l r t t (. . o t t t pr r o pr or t l ttl po bl).

ort t l t lo l pr or r t to lob ll t l r p o l or t l o plo lob l r p l or l l t b t r p r o rt t r p t t t r bl n-bo probl . r p l or b t o - j t rt o ot lo t pr t b t r r l p rt o pr - b r l or t to o l r t t t t oll p o t l o r.

p rt l r r t o or - r t pl t t t b o l or t b r t r ol ( ) 7] t l r t o o or l l or t . ro t po t o o t lt l l ppro t t- tr t t r t l t r t to o r r pr o l l l t t l l o t. b r o o- t o b o o r p r t t p rt l r b o t t o l probl o t t r r l r r p . p pr l ll r pt o o t l or t r b t r b t pl t t o ll]. pr pl o r t o l b po bl to t r t - r t l l or t or t p rt o t lt l l r p r lt o pr t r t l or t b o t t r q r t .

• ••••••• ••••• ••••••• k. r l p rt o t l or t t o o t t r l pr l t k_l (t l t t pr t r t or o pr). or t t l o r t r p G_L t 2 rt r pl t r o k_L t to b t t b t t . o r t t t rt o t t o o t pl t l or t or r p G_l $(l < L)$ t rt ll ll b po t o t r b t l o t l l t or r p G_l . t t r or o o t k_l r l t to t t l o t or r ot to tro t. k_l too l r t t t r r p ll to p ro t rr t l o t pot t ll r t t lt l l pro .

t r t l or k b o r t o r r p G_l t ll pl rt (. . ll rt r ppro t l t t k ro ot r)] j t o r o o k_l $\sqrt{\ /7}$ k_l . r bl t pl or l or r rob tl o r ll t pl t t t t lt o l t t t p r t r o l o t rt r t t o .

•• •••••••• •••• ••••••

ort t l t o pl t o t l or t or t r t o o r p $G_l(V_l, E_l)$ $O(\ V_l\ +\ E_l\ )$. or t t p o p r r p t t r t r t t V_l o po t l o t t pr o t r or t r r t or r t r -t . r ot t o t t lo t r p l or r tl ll o to b l t . t R to b t t o r r p l or ll t

t o t l or t b or lob l or b t p r o rt rt r p rt t R (t tl pl t p r po r to o l l t t t b t r p r o rt).
t or l l or t t l R $2k$ b t or t l r r r p t t r t r t t ot pro t to ‘ t l t lob ll . ort t l t l r r t l to R t lo r t l or t t to r o lt o R 2 k b tt r r lt t o t t p lt . ort t l o r t po r o t lt l l p r o to o r o R to o t l l l. or t t l o r r p t R_l to b r l t l l r o pr t l t o t too o t (V r ll or t r p). l or t l l r r p o t o t lob l t l lr b R_l r l t l ll t o t p l t pl t. t t r t l t t tr t R_l $2(l+\ )k_l$ or r p G_l or o ll t t ot p r t rt r. l o r pl t t o o R $2k$ or G t or l l or t .

t ot to r o pl t r lt or t l or t b t t t o bo . r tl t b r o r p l l L p t o t r t o o r . t b t t b r o rt ll b r b tor o 2 t r l l (t o t r rt t ot r o) t or t t o o l t rt t r l l (. . t r p t r r p ‘ b rt o t to r ot r rt o o l o t to t b). lo V $L < V$.
t t t t l or t ot ll t to r t r t p r p t or t pl to 3 t o r r t lo to 2.
t o r p rt o t l or t r $O(V_l + E_l)$ or l l l b t t t tot l r t l o t b t l or t . t bo pl t o (2.3) o l t lo r r p l or t t t r t o o t l or t bo b lo b $O(V_l + E_l)$ lt o t l r o t. t t r p r or t or t o pl t r p t b t t t t ll $O(V_l + E_l)$ p t o t r l t b l o ttr t r p l or . o r p t t t o l or t ppropr t or r r p (b t l r t t orr po to t tl p ‘blob).
b r o t r t o t r l l t r b t ool l t t^i t t p r t r t t r t o to t^i $(\ -\epsilon)t^{i-}$. t p r t b lo ( ]) t l t p r t r t k_l ϵ . t l or t to o r ll o t l t . k_l t t ll o t t t r t o i r $(\ -\ .)^i <$. or ot r or t r t r t o .
r t tot l o pl t t l l lo to $O(V_l + E_l)$ or p r r p t r t l o t b t t r t o .

ll o r t l or t t o t o r o
p r r p o N (. . t r l -l l pl t ( )) o p r
t t lt l l pl t ( ) o t r p . t T_p b t t
or t l or t to r o t r p or l t T_c b t t
to o r o tr t t. ppo t t t o r r t lo to
2 (tr or t pl b lo) t or t r
o probl o $N, N/2, \ldots, N/N$ l t t (l o t) l r o pl t or
t tot l r t or $T_c + T_p/N + \ldots + T_p/2 + T_p$. ll t
pl t t T_c T_p o l t t tot l r t
o ppro t l $T_p/N + \ldots + T_p/2 + T_p$ $2T_p$. ot r or o l
t t lo to r (t t t pl b lo
3. r b tt r r lt). t t l l l o t l or t
l l to lr r oo t l l o t t t t o l
r t r t lt o t tr l o t b t t t t
t o r r t or ll o t l t 2. o t l t tor
o 2 oo ‘r l o t b ot t t t o l or t r
$O(N\ )$ or $O(N\ )$ t l r l t t t t r t
b t t ll t t t t o .

pl t t l or t r b r t t r or o
p rt t o o t r tool lop t r . p-
r t r rr o t o ltr t 333
2 b t o or .

t t o r lt l l l or t o b r o pl r p
l o or lr o oo l o t (lt o t r t l
t r lt o t r p pro r ll t). o t r p
r r ro pl o ro r o o p t t o l -
probl . p ll r p t rt t r r pr t
o (t o l r p) or l t (t l r p). o r
l o o r r p ro ot r o -b ppl t o .

bl r o o pl o t r (V E)
t r r o t rt t r t
ort r pt o . ll t l or t pro oo l o t
lt o o t t rt o t o b l ( ]). o r or
o o t pl t l -l l l or t bl to l o t
t l t r p lob l (pt j t t p r t r
llo o r bl lo r r t). or t ttl 32
r p t l o t l l t 3 t r t l ttl r
t 2 l o t (. . o p r t 32 97 r t).

o l o t t bl o t r lt or r o r l
l r r r o o t (r t b pt r t)
to o t l b l t o t l or t (lb t o p r rl

• • • •• • • umm th mpl g ph

g ph	|•		|•		m m	g (c .)		u t m g ph t p
c t	6627			6.	2. 3	m cl qu t t
6	6	72	3	2. 3	. 2	u l
huttl	2	3	7 3	.	2 . 3	l
32	6	62	3	3. 2	.	l ct c c cu t
7	7	7	3 2	2.	3. 3 2	u l
h t k 3	2		3	. 2	2 . 2	l
2	7 7 2 26	2	2	6.	6 . 6 l	p g mm g
p k	7	77 7	2	.	2 7. 2	l m l ct l
m h	3	2 76	2	3.	2 . 3 3	u l
pl c .	237 7	3 2	3 2	2. 7	6 . 6 2	u l
pl c .2		6 7 3	3 2	2. 7	. 2	u l
pl c .	7 3	3232	3 2	2.	2 7. 2 2	u l
pl c .	76	277	3 2	2.	77.36 2	u l
pl c .	22	3 336 2	3 2	2.	. 7 2	u l

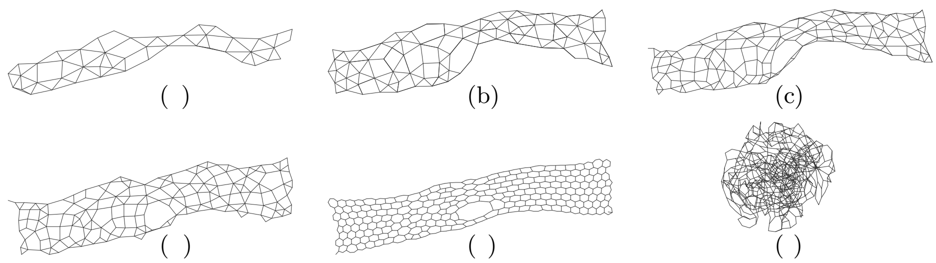
o o o r p). pport t o pl t l 2. t t t r t ppro t l l r $V + E$.

• •• • • • • •• • • • • • • • • • ••

t t o o tr t l ttl or t l o t lt l l pl - t ( ) l or t or or t r p . l or t r t o r t probl (r t b r o rt b tor o ro . t l l) o tr t r o 9 r p o . t l l o t o p t b pl t 2 rt o G t r o tt t t r l pr l t k to b t t b t t . t rt ro G_l G t l o t t rpol t ro G_l b pl pl rt t t po t o t l t r r pr t t t o r r r p t r .

r ( ) o t l l o t o G lt o o r t ll r t t or l t l o t lr b to t p . r (b)- ( ) l ll tr t t pl t l or t o G . r (b) o t t l l o t l l t o G t o t rt o t l t r ( ) t o t l o t t r t r t t r t o r t o t rt t rt to p r t . r ( ) ll o t l o t t r t pl t l or t o r or G . ot port t t r o t lt l l pro o o t p rt t o t t o l l t ll o t (p rt t o) o ot r r tl ro t t lo . r ( ) o t l l o t o t or l r p G . t r r t o t l or t to o p t t l o t j t o r o .

or o p r o r ( ) o t l -l l l or t o r o t l l o t. o bl t l or t ot ll t or t pro-



• •• • • • h mult l l c ct pl c m t llu t t th m h 6

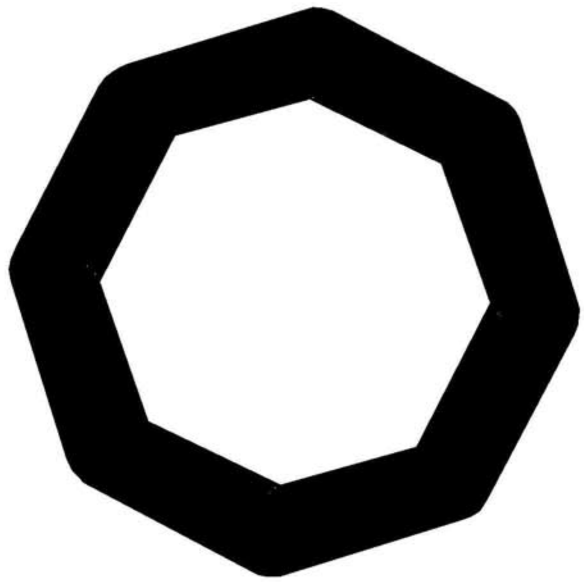
bl (lt o t t l t p r t r r to k) b t t l r t t lt o t ro tr t r b r o tr t r o bl ll t l- l l pl t ot b bl to ‘ t l t r p lob l .

• •• • • • • • •• • • • • • •

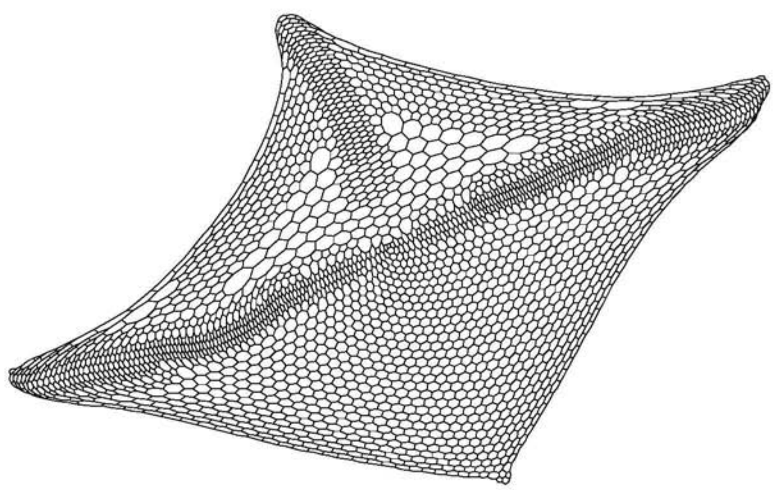
p pr l t pr t to o t pl bl b t - r 2-7 o o o t l t (l o]). p t t to t t t l or t b t t to p r r p (2.) r 2 o r l r r p - t - (r t to t t l or t or t l- q probl) o tr t t t t l o t l pt r t tr . l r 3 o t l o t l l t or t 97 r p b t l or t r tr to q l t l t t r t ll r l r or o t r p t t or l l o t. r o t 3 l o t o t ttl r p r l 3 l o t ‘p l t t o ot t t or l l o t. l r o t l o t l l t or l r rp r p l- l r ‘ r t l t p tr t r l r ol . ll t tr -p o] r o t l o t o b t l or t o 2 l r pro r tr t ro 7 rt . t l o t l ll - t t r p r l to rl r l r tr t r o t o r t 32 ‘ l o b r o ro protr . r 7 o t l o o o t l .

•

r b lt l l l or t or or - r t r p r o r t r p r t l o t t l l t t rpol t t r - lt o to t t l l o . l or t t . . bo t o or 2 l o t o rt p r r p bo t t or 22 rt - . or r o t t r t r t pl t t o o or - r t pl t (. . ro 7 o or rt r p 3])



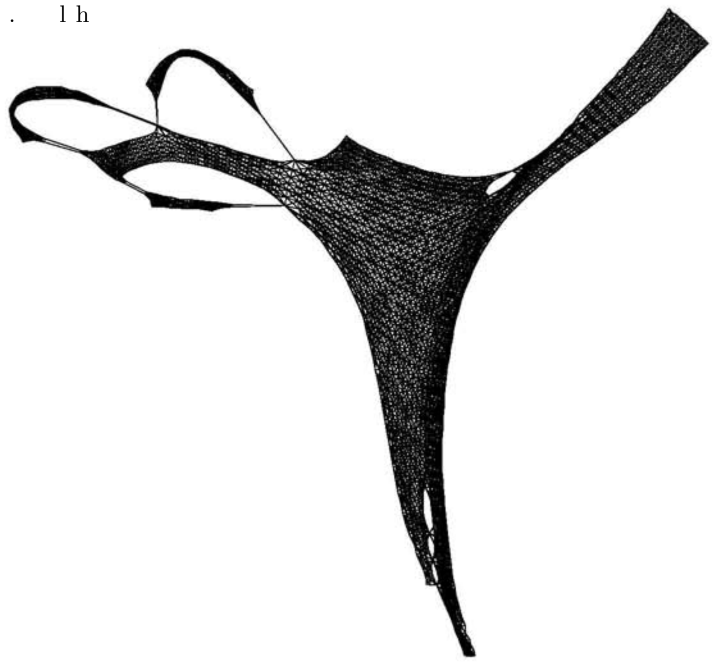
• •• • • • h l ut c t c mput th th mult l l pl c m t lg thm



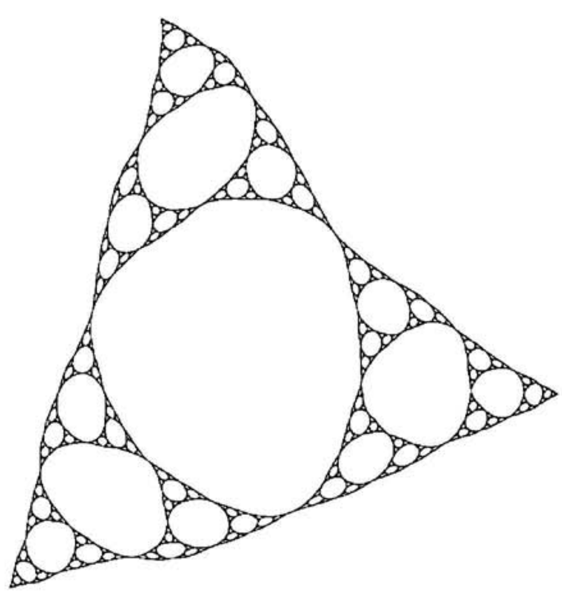
• •• • • • h l ut 7 c mput th th mult l l pl c m t lg thm

t ot l r t r rr t l -l l l or t pro-
r o bl l o t or l r r p . t lt l l p r
bro t op o or - r t l or t .

. l h

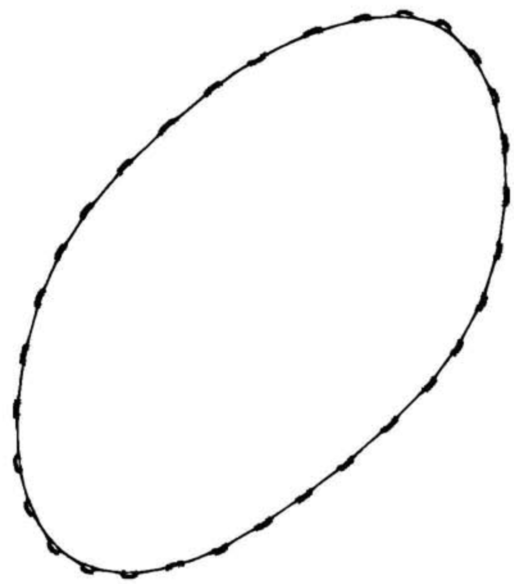


• ••••• h l ut huttl c mput th th mult l l pl c m t lg thm

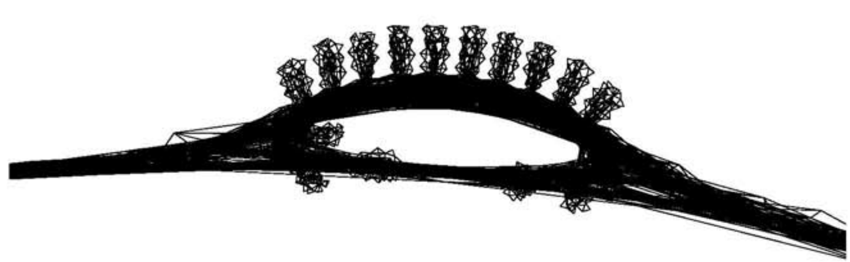


• ••••• h l ut p k c mput th th mult l l pl c m t lg thm

ult l l lg thm c ct ph g



• •• • • • h l ut 2 c mput th th mult l l pl c m t lg thm



• •• • • • 2 c mput th th mult l l pl c m t lg thm t l th m c t uctu

ot p rt l rl tr to r r p or t t q t ot or . t l l t t r r p or t o b tr t r r r o to b oo t or l or t o r o pt o . t l o l l t t r p o t rt o r r ot p rt l rl t t o r pro (2.). o r b l t t t lt l l pro l r t l or t or r o l r p rt r t t o r t t p o r p port t bj t or rt r r r . ot r o t r p b t l t t t r q r o l or o t o .

. . . l. g ph c l u g mul t l g. ( ) 3 33 6.
2. . tt t . . m . . ll . . t c ll . . . .
3. . . u c . . ch . . b u . l t g lu t g mb g g l ph . . t ch l t lum 73 . p g .
. . . u t c ph g. 2 6 .
. . . g. ult l l u l t lu t ph . lum p g 2. p g 6.
6. . . g . . g m ch . t ght g lg thm ch c l ph lu t ph . ch. p. 3 pt. mp. c . . c tl ll gh 23 u t l .
7. . . . ucht m . . g l . ph g b c ct l c m t. 2 ( ) 2 6 .
. . . l. ult c l lg thm g ph c l . ch. p. m t. c . cult th . mp. c . .
. . l . . t ult c l lg thm g g ph . ch. p. 2 m t. c . cult th . mp. c . .
. . ck . l . ult l l lg thm t t g ph . . t . .
. . . . p m t u . t gl t . . t c ll gl l 2.
2. . bl k . ck. ut m t c ph lu t g. lum p g 3 . p g 6.
3. . u k l g. t ct l ph ut m t. . . h t t lum 7 p g 3 22. p g .
. . l h . ult l l lg thm c ct ph g. ch. p. / /6 . ch p l 2 .
. . l h . ult l l pp ch t th ll g l m bl m. ch. p. / /63 . ch ug. 2 .

vi r l u or

pt. o pp i t tic o put ci c
i titut o ci c o ot
{harel,yehuda}@wisdom.weizmann.ac.il

io pp i titut c . po t 99-2 999.
i io pp i *s t s*
2 2 pp. 2 2-2

p t u ti- c out o it o t t tic
i o u i ct p it t i t- i . o it i
t t i c p o i p o u t ti -
i t ot o it o . o p t
o it c i opti i o t p i out 2
co . p p co t i p it o 6 o . p opo-
o it o i u ti- c c o i p
ic oti t t c t pu i u ti- c o it o
]. t c i i c t i p o t p o -
ti o c - i ct t o (o t t t o ’ i it o
i i t p o t co ti uit o it co t- u ctio).

r p $G(V,E)$ is s r s ru ur is us o o l r l io E ov r
s V o i i s. r p r i is o v io l ool or visu li io
o r l io l i or io i s us ul ss p s o i s r ili is
p ili o o v i i o i r qui kl l rl .
r rs l ori s or r i r p s u o i ll r propos
(s o r is surv o pr siv l i 2]).
o r o pro l o r i u ir r p i s r i
li s. is s pro l r u s o o posi io i v r i s
r i i ppi L $V -$ $\mathbb{R}$. popul r ri ppro o is
pro l is *f* *t* iqu i i ro u s uris i os u io
( ) o ppi L i (op ull) i v s i s i i u
l ou is i . rious v ri s o is ppro i r i i io o
r i op i i io o s i s i i u . o k o
l ori s r os o 4]] 3]]. jor v s o or ir
o s r ir r l iv l si pl i pl io ir fl i ili (uris i
i prov s r sil) u r r so pro l s i oo.
s v r pro l is i ul o i i i i r li i

○

. . o

l r r p s. ov o s o us o r p s o up o v r i s. or l r r r p s o v r o i i u i possi l ll is v r slo .
propos o or r i r p s si i l i prov sp o v r or ir o . uil our l ori rou i o r sul i l ori i is r l s is p l o r i r p s o su s i ll l r r si o r l ori r r o . l ori i s o iv r ul i s l l ori o r l 7] orks pro u i s qu o i prov ppro i io s o l l ou . ppro i io llo s v r i s o vi ro ir l pl li i r si o s r. s r sul l ou o pu usi i r si l o rs r pr s io s o r p r los l r v r i s r oll ps i o si l v r .
l ou i s qu is r v r r pi l p r or i lo l u i io o pr viousl r l ou .

i ui io o 7] or u i r p l ou is r p s oul i o ll s l s. o r or s r i s oul i o i ro l v l ro l v l. l i o l o is i ui io ill or li o io o r l v o r p r i pro l . ru i l o s rv io is lo l s i s r r o p o r r l o l r r s o pi ur is r r i i s i ro s ru ur i s o l i or i p o lo l issu o u . o r lo l s i s r r o p o r li i o s ll r s o r i . ollo i is li o i ki ill o s ru o r i s ri ki o s r r los o o r i o si l o o i i r i li i s lo l ils u pr s rv s lo l s ru ur o ori i l r i .
l r iv vi o our o io o o rs i is s ppro i io o i l ou . is ppro i io llo s v r i s o vi ro ir l posi io ou li i o so o s r. s o s qu u i ll v r i s os l lo io li s i i ir l o r ius r us o i o rs s l r pr s io .
ur pr s io o r i s is pr so i io s

l ou f $G(V,E)$ f t t t t
L_G $V -$ $\mathbb{R}$ ft t t t G
t t t t t t t t t
t f t t t t f t t t t
t i t f G t L_G L

L lo ll i t f $G(V,E)$ t t t r f t t t f $L(V)$
t f r t f t t f
G

t u ti- c t o o i p

L lo ll i t $f\,G(V,E)$ t $t\ t$ r f

$$_{v\ \ V}\ L(v) - L\ (v) \quad < r$$

lo li pr s rvi k lus ri k lp f t $f\,G(V,E)$ t $t\ t$ r t t $G(\ V\ ,V\ ,\dots,V_k\ ,E\ ,w)$

V V $V\ \dots V_k,$ i j V_i V_j

E (V_i,V_j) (v_i,v_j) E v_i V_i v_j V_j

$$w(V_i,V_j) \quad \frac{}{V_i\ \ V_j} \sum_{v\ \ V\cdot,u\ \ V\cdot} d_{vu}\,, \qquad V\cdot,V\cdot \quad E'$$

d_{vu} t t t t t G

f i

$$_{v,u\ \ V\cdot}\ L\ (v) - L\ (u) \quad < r$$

t t t t t t t t t f k lus rs

rs l is i io s s o o li l pr i l v lu s i r rs o u k o i l ou L . ill is uss is i por poi i s io .

ul i s l r pr s io f $G(V,E)$ f $G^{k\cdot}$, $G^{k\cdot}$, … , $G^{k\cdot}$ $k\ < k\ <\ \ < k_l$ V f $\leqslant i \leqslant l$ $G^{k\cdot}$ t k_i t $f\,G(V,E)$ t $t\ t$ r_i $r\ > r\ >\ \ > r_l$

ur ll ssu i i l ou o i r p l s o s v o r fl ir i s.

ur o r li s o s o r i r p s i s ll u r o v r i s o ollo i o ssu p io s i or li i k o ili o ul i s l s i s i p lo l lo l s i s.

$t\ G_r$ k f G t $t\ t$ r $t\ r \geqslant r$ L t $f\ G_r$ t $t\ t$ r f $f\ L$ t $f\ G$ t $t\ t$ r G t $L(v)$ $L(V_i)$ f v V_i

L is i l ou o k lp o r p G i r sp o r
L is lo ll i l ou o G i r sp o r.

i ui io o ssu p io 2 is lo l s i s is i p o
i ro s ru ur o r p so i r s l ou s o G_r
o G r ou i r.

f L t t f G t t t r
t t t f G

o pr s ul i s l r i s i r s r p
pro u i s qu o i prov ppro i io s o l l ou .

t

. l v r i s o G r o l i r i r .
2. oos qu r si s qu o r ius s $r > r > r >$
$> r_l$.
3. *f i t l*
   3. oos ppropri v lu o k_i o s ru $G^{k\cdot}$ k_i lp o G .r. . r_i.
   3.2 l v r o $G^{k\cdot}$ (i) lo io o v r i s o G o s i u i .
   3.3 o ll u i lo l i or oo s o $G^{k\cdot}$.
   3.4 l v r o lo io o i s lus r (i. . v r i $G^{k\cdot}$).
4.

vi ili o s s s ro ollo i o s rv io s

. rs i r io r s p 3.3 s oul v i l ou o $G^{k\cdot}$. o u r is v lu o r s o l r ou so r sul i $G^{k\cdot}$ ill s ll sil r i l .
2. p 3.3 s oul i l lo ll i l ou o $G^{k\cdot}$ .r. . r_{i-} . o u r is v o oos l r ou i or oo . o oo l r o v r us o r i op i ll s r o . su i l s oi o s qu o r_i s ill o.
3. i i o i r io i v lo ll i l ou o $G^{k\cdot}$ .r. . r_{i-} . ssu p io 22 r s p 3.4 v i l ou o $G^{k\cdot}$. is l ou is lo ll i l ou o $G^{k\cdot\cdot}$ .r. . r_i ssu p io 2 .

r rk oi o ul i s l r pr s io s
upo i r r si s qu r , . . . , r_l (s s ri ov) or
i r si s qu k , . . . , k_l V (s v o i our i pl io).
i io 24 i s o s o k lp i or r o r
i si propor io s o i is ss r or ki ssu p io 2

v li . o v r i pr i o j ur our s orks ll v i ou i i s o k lp usi so v ri s o spri r o (. . os o 4]]) s lo l i io o . r so or is is su o s si i l ro r i i i li io i il o p uri l si o r p o solv s l r s l o fli s orr l .

• • • • • • • • •• • • ••• • •

or r o i pl ul i s l r p r i l ori s o s o io 2 v o ur r l or o o poi s ( ) o o ul i s l r pr s io o r p (li 3. o s) (2) o o vis lo ll i l ou (li 3.3 o s).

or r o o s ru ul i s l r pr s io o r p G s o i io 2 us k lp o G i i v r i s r r los i i l ou s oul roup o r. i por qu s io is t t t f t t t t t u kil o v uris i lp i i v r i s ill r los l . or ov r is k isio v r r pi l is jor r so or s ru i i o our l ori . uris i is s o o s rv io i l ou o r p s oul o v visu ll r l io l i or io r p r pr s so t t t t t t t t t t . is uris i is v r o s rv iv ll or ir r i l ori s us i vil . plo i is uris i ppro i k lp o G usi l ori or ll k o k t pro l . is pro l is o p r i io V i o k lus rs so lo s r p or i is o v r i s i s lus r is i i i . r li oul lik o i i v r v r i lus r i si l v r ppro i s r r o lus r. us solu io o los l r l k t pro l r o oos k v r i s o V su lo s is ro V o s k rs is i i i . s u l pro l s ris i r s v i l i v s i i s v r l p p rs (s . . 6] 9]). or u l o pro l s r r i s s o i 6] 9] u l ss r o s o is $(2-\epsilon)$ ppro i io l ori or $\epsilon >$. v r l ss r r v rious s si pl 2 ppro i io l ori s or s pro l s.

• t i i o u ci t (i ot c) co itio t t tic co .

• δ- pp o i tio o it i pp o i t o utio u t to it i co t t cto δ o t opti o utio .

. . o

ill ppro i k lp s solu io o k r pro l op i 2 ppro i io o io i]

$(G(V,E)$ $k)$

i s S V o si k su v V i s S d_{sv} is i i i .

. S v or so r i r r v V
2. f i 2 t k
2. i v r u r s ro S
(i. . su i s S d_{us} $\geqslant$ i s S d_{ws} , w V)
2.2 S S u
3. t S
4.

t i 2. rri ou i i $\Theta(\ E\ )$. our s i o s r si s s ll s v ll p irs s or s is our s (i is or lo l u i io). ili i is ori i urr is o v r v r ro S i pl li 2. i i $\Theta(\ V\ )$ i l i o l i o pl i o $\Theta(k\ V\ )$.

v os o us v ri o i o] s our lo l r i o . ou i o v r ppropri us i r l s v r p ir o v r i s so o s ru i o rs r pr s io o r p o o v o i p irs o v r i s r o . o i is prop r o i o s pri i or s us o s $\Theta(\ V\ )$ or v r p is sp rs . o r v o i o is i l ir l i i r p s i is o v i i our s si ul i s l r pr s io o r p o i s i r p s.

t o si r r p $G(V,E)$ r v r v is pp l ou L i o poi i pl $L(v)$ i oor i s (x_v, y_v). is d_{uv} is s l o s or s p i G u v. k o v o $N^k(v)$ u V $\leqslant d_{uv} < k$. or r o l ou i s i l pl si k i or oo s us r u io l r l s r p or i is v r i s i r p o u li is i r i is s ollo s

$$E_k \quad \sum_{v \ V} \sum_{u \ N^{\cdot} \ v} k_{uv}(\ L(u) - L(v) \ - l d_{uv})$$

r l is l o si l $L(u) - L(v)$ is u li is $L(u)$ $L(v)$ k_{uv} is i i o s i r $\overline{d_{\cdot\cdot}}$ or $\overline{d_{\cdot\cdot}}$.

r r pr s s or li squ r rror u li is o v r i s i pi ur r p or i is . l p irs i s k i or oo r o si r .

t f t ur purpos is o l ou ri s r E_k o lo l i i u . ss r o i io o lo l i i u is s ollo s

$$\frac{\partial E_k}{\partial x_v} \quad \frac{\partial E_k}{\partial y_v} \quad , \quad v \quad V$$

o i v is o i io i r iv l oos v r s l r s v lu o Δ_v i is s

$$\Delta_v \quad \sqrt{(\frac{\partial E_k}{\partial x_v}) \; + (\frac{\partial E_k}{\partial y_v})}$$

ov is v r v ou o (δ_x^v, δ_y^v). o pu io o (δ_x^v, δ_y^v) is rri ou vi i E_k s u io o o l $L(v)$ (x_v, y_v) us o o i sio l o p so o . s r sul u k o s δ_x^v δ_y^v r ou solvi ollo i p ir o li r qu io s

$$\begin{aligned} \frac{\partial \; E_k}{\partial x_v} \delta_x^v + \frac{\partial \; E_k}{\partial x_v \partial y_v} \delta_y^v \quad & -\frac{\partial E_k}{\partial x_v} \\ \frac{\partial \; E_k}{\partial y_v \partial x_v} \delta_x^v + \frac{\partial \; E_k}{\partial y_v} \delta_y^v \quad & -\frac{\partial E_k}{\partial y_v} \end{aligned}$$

i r s r r ur r ils i].

t f *t*

ollo i l ori i is ro] s io i] is o i io o]. l ori o pu s i l ou o v r k i or oo o r p

(d$_V$ $_V$ L)

i lo ll i l ou L u i i k i or oo s

. *f i t Iterations V*

. oos v r v i i l Δ_v

.2 o pu δ_x^v δ_y^v solvi ov io qu io s

.3 $L(v)$ $L(v) + (\delta_x^v, \delta_y^v)$

2.

pi l v lu o p r r *Iterations* is 4. (ru i is l o ri s s lo (o s p r o ul i s l l ori) v lu s oul l s .)

t o pu io i o s p . is $\Theta(V)$ si ori rs riv iv s o E_k up i i $\Theta(N^k(v))$ r ov o v r v. o pu io i o δ_x^v δ_y^v is $\Theta(N^k(v))$. s o pu io s r rri ou *Iterations* V i s so ov r ll i o pl i is $\Theta(V)$. s l v r v i i l Δ_v i o s i ki is s l io rou s pli o s si su s s o V i ou s rious r o qu li o r sul s. or ov r i r o G is o u $\Theta(N^k(v))$ is o s . s r sul ov r ll i o pl i i ou r s is $\Theta(V)$.

ull v rsio o p p r s ri l r iv u i io o i li i s oi o v r i i l Δ_v.

o s ri ull l ori

$(G(V,E))$
i L i l ou o G

. o pu ll p irs s or s is $(d_{V\ \ V})$
2. up r o l ou L
3. k *Threshold*
4. $k \leqslant V$
 4. *Centers* $(G(V,E)\ \ k)$
 4.2 $(d_{Centers\ \ Centers}\ \ L(Centers))$
 4.3 f v r v V
  4.3. $L(v)$ $L(center(v)) + \xi$
 4.4 k k *Ratio*
.

t li 4.3. ssu ll *center*(v) r ur s r is los s o v. s ll r o ois (,) $< \xi <$ (,) us our lo l u i io l ori p r or s l v r i s r i i i li o s poi . *Threshold* *Ratio* r o s s i pi l v lu s o 3 r sp iv l . i r o i prov lo l s i s i r iv l r p i li s 4. 4.3 u r o i s r s s i k (. . $\frac{V}{k}$ i s).

t ov r ll s p o i l o pl i is r i o pu io o ll p irs s or s is (li) i i pl i i i i ro v r v r . us k s i $\Theta(V\ E)$. or v r sp rs r p s lo l u i io (li 4.2) o su s 9 % o ru i i o pu io o ll p irs s or s is o su s r i i 2 %. ll o r p r s o l ori o su o l li i l i . s si o r p s o s l r r o pu io o ll p irs s or s is o s or o i . sp o pl i is $\Theta(V)$ si v o ori ll p irs s or s is ri .

• • • • • • ••• •

is s io o i s pl s o r sul s o our l ori . i pl io is i ++ ru s o iu 33 . s pro ur o u i i or oo s o r ius 7 *appr* r *appr* is p pro i is i or v r i s i urr r pr s io . o s *iterations* s s o 4. pi l u io i i s p r rs is ou 2s or v r r p s u r 2s or 3 v r r p s 6 s or 6 v r r p s. ll op i i o ill pro l o r i 6 v r s .

s s l ou s r r l ur l looki ; l os "op i l i s i s. vir u ll p r l ou s s o i i urs r pi l o po r o l ori . r p s i i urs 6 7 r p r i ul rl i pr ssiv s " orr (ri or orus) pp r is r i spi p r i li o i or io v il l o l ori s.

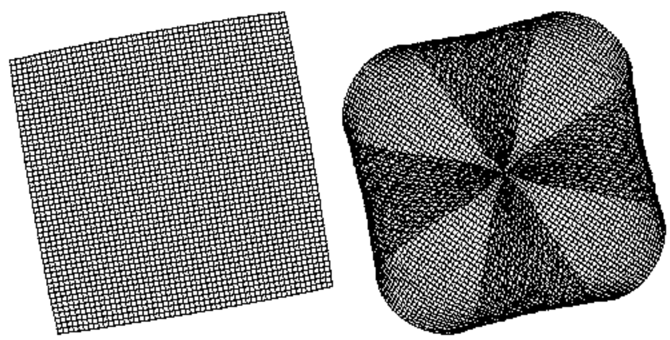
r i ull i r r s i i ur ook or or l i . or ir o s v r o i o r p s si r r or s ork i s ir io s. p r or qui l o r s. ur o su i i i l ou o r s o l i o si r ir r p s o i i s "lo l u i io s s. pro l is lo l u i io s is s o l o i or oo s i r p or i s s u i p ki s o r s l v s s oul s o up los o o r v i r r p r i r p or i s s .

io v i l i ou ull i r r s our o s o v s o p r o is o]. irs i s ru i i is s ill u s r (ou s or 23 v r r) o ou o r l iv l u i io i r io s. o r sul i pi ur is l os pl r s o lo l is or io s i o r s o is o i ill r pp i lo l i i ill l s r sul i rossi s.

i ur 9 i is ri i so o ori o l s r ov illus r s si il r pro l . i ur 9( ) s i s up so lo l u i io pro ur o si r l r r r ul r i or oo s s is is o l o k i o ou o r v r i s o o o s u iv "li s o k r sul i pi ur pl r. o r si i i ur 9( ) lo l u i io pro ur o si r i or oo s r oo s ll ov rl ppi .

is i r s i o io i ur 9( ) i volv s or lo l o si r io s ki i o ou l r r i or oo s is o o l v . s lo l o si r io s so i s o p s o i por lo l s i s r sul i i ov r ro lus rs o v r i s. r l ul i s l s i s v u l v ov r r ul r s i s i s p r lo l o si r io s ro lo l o s. ul i s l l ou l ori s rv s o o l s o i i i o pl r u io r pi l u o lso s r r l ppro o r p r i . i i lo l lo l s i s is op

o is i uis pri i iv lus rs o r p r illi o p rl i i or o si ri l r r lus rs. or pl o si r li r o i ur . i ur ( ) u i io pro ur o si r i or oo s o r ius 4 i i ur ( ) l r r i or oo s r o si r . us orr ul i s l s ru ur o r p s i i i ur ( ) is sup rior o i ur ( ) i i s ru ur o s ll ir l s is o pp r .



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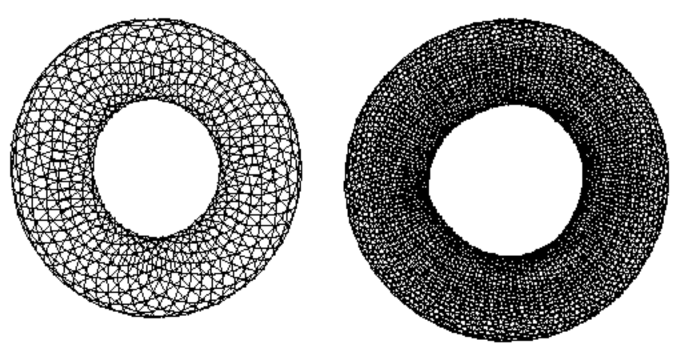

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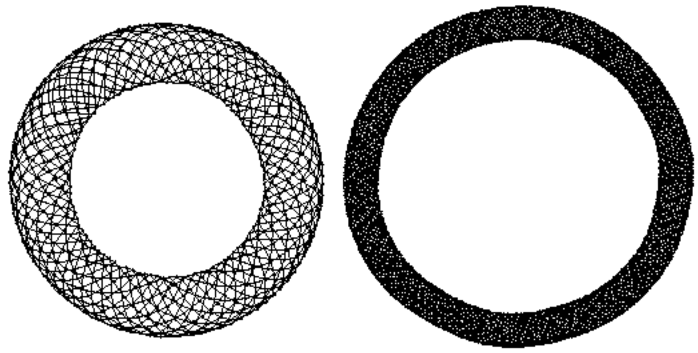

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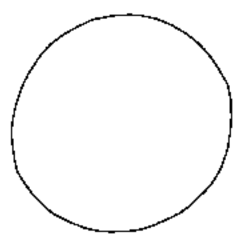

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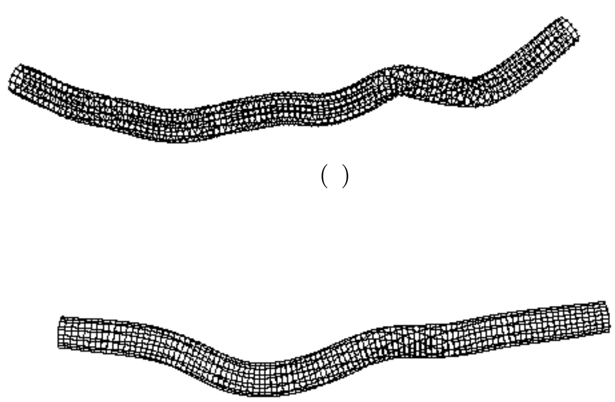

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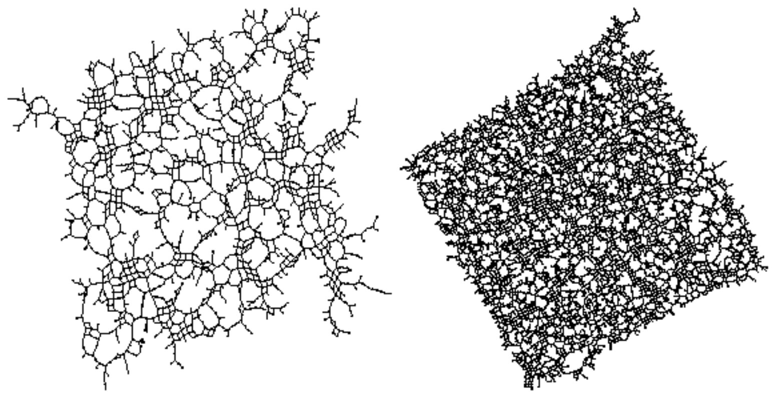
• •• • • • (- t) c i uti c tio co i i o oo o - iu i ( ) i ( )



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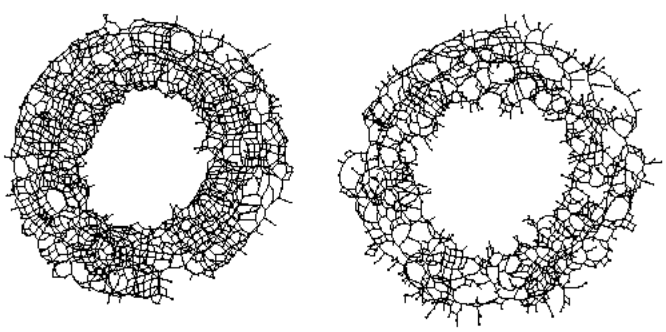
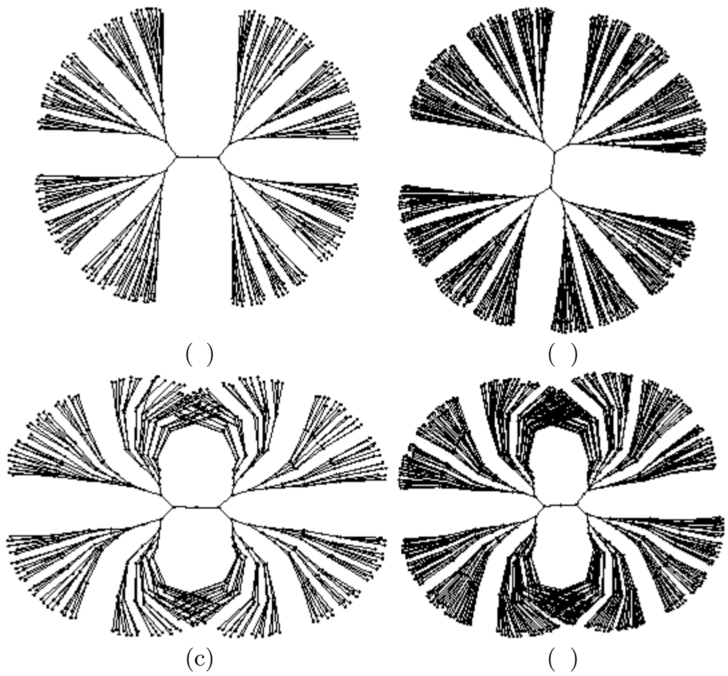


( ) ( )

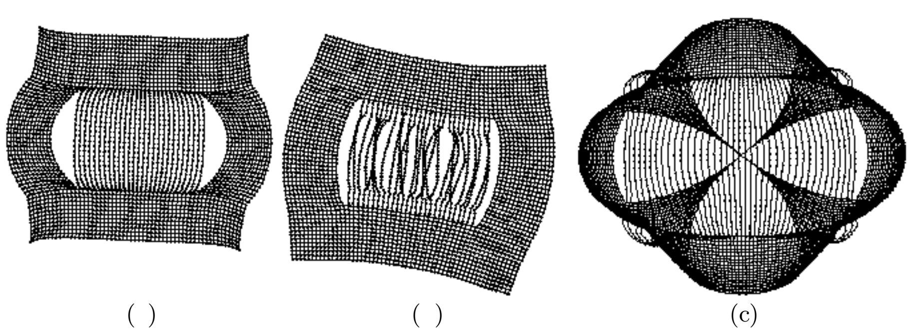
2 (6 - t) to u it ÷ ÷ o t o itt t o

v pr s ul i s l ppro or r i r p s i l v su s us ul or ul io or sir prop r i s o o rs i s. ur l ori is l o l r l ll r l s i l r r p s. l ori s si or sp si pli i o s o r quir pli i r pr s io s o o rs r p s.

or po r ul r l i pl io s o ul i s l r i s ill ov r o so li i io s o our l ori . l ori o r quir o l li r i sp is r i ll p irs s or s is o pu io (o r l i or o i o). is ill l l ori o l i r p s o ov r v r i s. o



• •• • • • u i t (c) - tic pt ( ) 23 tic pt 9 uti c tio co i i o oo o iu 6 6 p cti i ( ) ( ) t i i t o p .



• •• • • • ( ) (3 2 - t) p i ( ) uti c tio co i t o i o oo o iu 3 ( ) uti c tio co i u u i o oo o iu (c) (6 - t) p i it c t o oppo it co co c- t

r i prov li s i o s ru io o o rs s l r pr s io or i us si pl uris i . or r p s is uris i is u or so r p s i o ou . or pl o si r r p s i i i r. ur uris i il o su r p s si is s p ir o v r i s r rou l s i is us u l o is i uis i r lus rs. r r o r uris i s r or o pli i pl io o ul is l s is u r .

• ••••••••

. u . . i o t . o . p i t o p i o o o c - i ct o i p i o it ” *s* ctu ot i o put ci c o . 2 pp. 6 p i 99 .

2. i tti t . . i . o i . . *t s t s t s* tic - 999.

3. i o . . i p ic i i u t i ” *s s* •• (996) 3 33 .

. . u i tic o p i ” *ss s t* •• (9) 9 6 .

. uc t . . . i o . p i o c - i ct - c t” *t t* •• (99) 29 6 .

6. o . u t i to i i i t i u t - u t i t c ” *t t* •• (9) 293 3 6.

. . . u ti- c t o o i p ic ” *s t t t s* i p 2 . (o *2 t t s t ts t* (’99) ctu ot i o put ci c o . 66 pp. 262 2 p i 999.)

. oc u . . (.) *t t s s* u i i o p 996.

9. oc u . . o . i pp o c to pp o i tio - o it o ott c o ” *ss t* •• (9 6) 33 .

. . i . o it o i i ct - p ” *t ss tt s* •• (9 9) .

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o i l t

http://www.cs.newcastle.edu.au/~aquigley
p rt nt put r ci nc n t r ngin ring,
ni . c stl , ll gh n, 23 , ustr li

• •••••••• st lg rith (FADE) r th 2 r ing, g tric clust - ring n ultil l i ing l rg un ir ct gr phs is pr s nt . h lg rith is n t nsi n th rn s- ut hi r rchic l sp c c - p siti n th , hich inclu s g s n ultil l isu l str cti n. p r t th rigin l rc ir ct lg rith , th ti rh is • (• + • l g•) h r • n • r th nu rs n s n g s. h i pr nt is p ssi l sinc th c p siti n tr pr i s s st - tic t t r in th gr •••••••• t n n s ith ut plicitl c lcul ting th ist nc t n ch n . i r nt t p s r gul r c p siti n tr s r intr uc . h c p siti n tr ls r pr s nts hi r rchic l clust ring th n s, hich i pr s in gr ph th r tic s ns s th gr ph r ing ppr ch s l r n rg st t . in ll , th c p siti n tr pr i s ch nis t i th hi r rchic l clust ring n ri us l ls str cti n. rg r gr phs c n r pr s nt r c ncis l , n high r l l str cti n, ith r gr phics n scr n.

o - i t l o it o t o i to t i fl i ilit o i l t tio t i o t l t lt t i . o i t lo o o i t l o- it [9 36 . fl i ilit o t o i t o llo o ki o o t i t [6 3 3 3 . o o o o t o o t o i t t o i t i i ilit to l to l l . o l t o t i o t tio l o t o o ti o t i o o .

i i t o t l o it o t i o l i t o t o i t o . i l o it i i l o . o o o i i i l o o ot o lo i o l t i t o -to o i t tio t o to o i t t o i i l o t i tio . tio i t o t tio l o t o t o i t o llo l to i l-ti .

○

t t t l i t i t o i ot o
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li it o ot t il l o i o t tl o itio
q i kl [3 3 33 2 .
t o t i i o i ollo . tio 2 i t
o t o h h o t
l t to t i li tio o l . tio 3 o tli h h
. l o it FADE i i i tio
. tio i t o l t i o t - i
i o t i t i . i t tio o t
i tio 6. ll i tio o lt o o o t
l t i t .

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o ( ) o i t o titio ∪ ∪ ∪
. o t o t o . h h i t o o o i
l t o t t t o t [22 29 23 6 2 3 . l i i
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i i i i i to t o l t it o l o o li t .
t o o t o ti l t i oti t li tio i ll l
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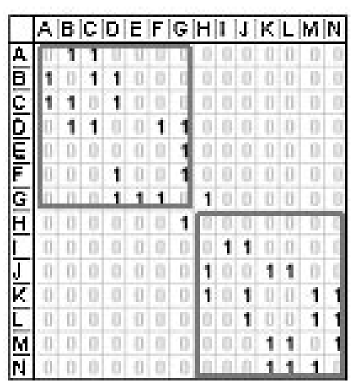

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
A	0	1	1	0	0	0	0	0	0	0	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0	0	0	0	0
C	1	1	0	1	0	0	0	0	0	0	0	0	0	0
D	0	1	1	0	0	1	1	0	0	0	0	0	0	0
E	0	0	0	0	0	0	1	0	0	0	0	0	0	0
F	0	0	0	1	0	0	1	0	0	0	0	0	0	0
G	0	0	0	1	1	1	0	1	0	0	0	0	0	0
H	0	0	0	0	0	0	1	0	0	0	0	0	0	0
I	0	0	0	0	0	0	0	0	1	1	0	0	0	0
J	0	0	0	0	0	0	0	1	0	0	1	1	0	0
K	0	0	0	0	0	0	0	1	0	1	0	0	1	1
L	0	0	0	0	0	0	0	0	0	1	0	0	1	1
M	0	0	0	0	0	0	0	0	0	0	1	1	0	1
N	0	0	0	0	0	0	0	0	0	0	1	1	1	0

• •• • • • r ph h r tic, singl link g clust ring pl

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lt i l t -lik l t ( ) it t t o ll
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l t i it i l t i o titio i o t
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. o t tio l o t i i o i t i l t i q i kl
o i t t o l .
2. t t o ti l t i i oo t t i o ti i o
o i t o .
3. lt t i i to t o t l t i
o t i l t it l i i li tio .

titio i o i [i o t o lti-
l l titio i [2 22 t i o t t ll
i titio t o to o t t titio i o t
o i i l . t o l ot t t t i t o i [o
i i o l t t i it oo t o ti l t . o
t it i ot t t o t- o i t o
t ti t o .

o t o oi t i - i io l i titio
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oi t t t t i i t l t . o t i l t i i i
i i t o t i i li tio t tt i t o tit
to oi t i - i io l o i t i t tio i
[32 6 2 3 . tio i t h i i o
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t it i o i t [2 .
i t o t i l t i i
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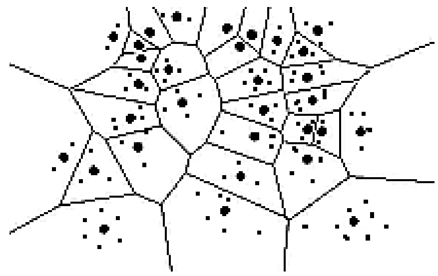
o io [2 . o t t t t o ti l t i i q lit t



• ••••• r n i i gr s n ps u n s, r l n s sh n s p ints

i t i it t o l tio i t o t i
t o ti i t t o .
i t o (FADE) t i o o itio
i i o t i l t i o t lo tio o t o t i i t
i t o ti l t i . i t o ti l t i i t
i o i t l o it t t i o t l tio t li
t o ti i t i t i t i i t i o t
t o ti l t i . t ti t i o i o ot t i t
l t i . i i t i o i t t tio l l o t
l t t i t ... • o t tio i t i t
o t o o itio t .

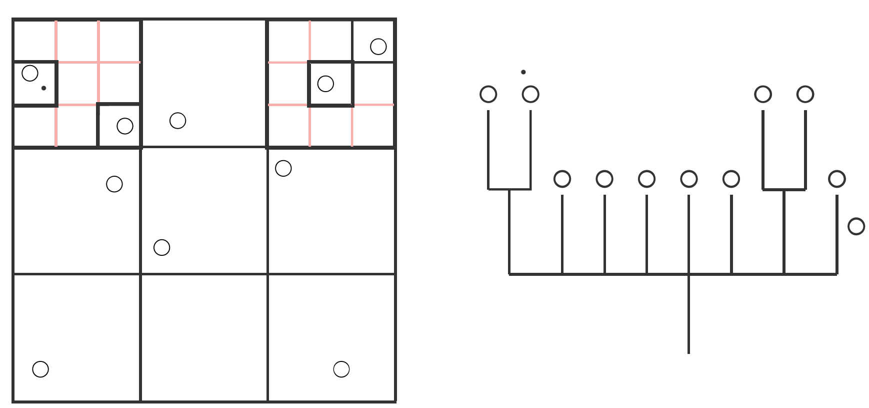
o li l o i t ti ti l i t t i i t o -
t i . i l l i l l ti l i l i l tio t
o t t o i fl i i [2 . i l i l tio
to t t i ll i ol - ti l .
l i l t t (o o) t o

••••• + •••• + •••••••• ( )

••••• i i l i tio o i t (t
l ot ti l i i l i) •••••••• i tio i i i t
o t l ti o itio o t ti l (ti l)
t o o t i ti ( ). o t o t tio o ••••
t ti l to ti l i t tio t k ti ( ). •••• o
t o t o o i t t o
(t [9) o i . o t o ( ) i i i o t
o i iti i ot . i t o - i t o o
i t li it tio l . i i t o
lo to t o t tio l o t i l l ti t lo ••••
o o o .
t o i ll l i i l i o
t i l tio t ti l o t i t t i to t o

it . o iti i t ti tio o t o o ti l ti l . o o l i ti l i l tio t i lti li it o - i o ti l i t i tio t t i t ti l i t i tio i l t [2 . o i o i t l o it o t i ol [ollo t o . t o o it l o o i ti t o i i i o ot i it i o o i t i tio .

t ti i i i io o i to i o ll i t t o i lit t [2 2 to - o o l l tio . o o i i i l ti l o ot ti l lo i o l t i t ti l -to- ti l i t tio t o to o i t t ti l i i l o o t i tio . o t lo i t o t t o l ti i l tio i ol i o illio o o i . o o l ( ) i l ( ) o o itio . (i - i o o itio) i ill t t i i 3. l t t i ilt i li ti .



n -tr sp c c p siti n n t structur

lik o il i o o itio t o ot o o - i o i t i tio o ti l .

o o itio o t o o i o t - o t o t i o t i l t i o t ti o t . - i l t i t -t o ot i tl o i o t i l t i o t o o itio o i it il i q lit

t o ti l t i . o t l t i o ilit t i o - t i t o o o i t l o it i o i tio . lo . t it llo o lti-l l i i o t io l l o t tio . i ll t q lit o t i i o (t t i it lo t t o t o t) t q lit o t l t i i it t o o itio t i o .

i i o t FADE l o it i t ollo i . i t o o itio t t o i i i l o o ot o lo i o l t i t o -to- o i t tio t o to o i t t o i i l o l t o t i tio . i i - it t t o ti l t o o l t l. [i t t o i t o o t l o t i i o tio t t o- o i t o . i i llo o o i t t o to li to l l it o t i to ti t lt t i to t l i t t .

i l t FADE i l o it o i t i t o i io i i l o o l q -t o o itio . o i t l o it t o i [6 3 3 9 36 o t t o ki o o ook ' l i o ti l io . i l i i li tio ( ) $\Omega($ $)$ o - i t o o . FADE l o it o t o - o o o o i t o - o i t - o o . t t i o t t o it tio o t o i t l o it FADE.

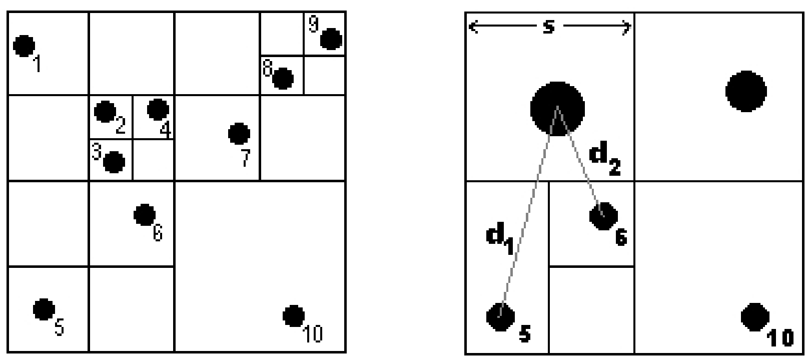
o l ki FADE o k ollo

```
REPEAT:
  Construct geometric clustering using a space decomposition
  Compute edge forces
  Compute nonedge forces (node-node and node-pseduonode)
  Move nodes
UNTIL convergence
```

t t t o i i o o itio q -t o i t ti to t i t o t o it o t li itl l l ti t i t t o o l i . o o t o i l l t i tl o i t t o o to t i to - o o o o .

o t i o t i i lo i [2 2 . t t ≤ i t i t o t ll i t i t t t t o t t o o t ll i t -tol - t . ≤ t t i t l o i o t o o t i tio o t o o i to t l ti o o t t o . t i t o o i ol i to it t o o (-t) o i i i l i . o o ol o ti i t - t t o t t til it t tol it io i ti o l o i .

i q -t o o itio l o . o t - tol t o l i o t o t o o t l t o i .



• •• • • • p ring n n 6 ith th rth- st ps u n , h r s/ . n s/ .46 r sp cti l

i t l o i o it t o ti l i t o t - t q . i t o t o o i t l ti i t o t l i it -t i t i . l i t i t o t ll t o o o i t i t t t o t o o . ÷ i t i . o t l . o t FADE l o it o t t o o t t i o o o it t t i o to t l ti o o o o .

o 6 i o it t o ti l i t i t t t o t o o i t o t - t q to t ÷ 6. t l o ot l ll t it io ≤ t o o i ol i to it - o (i t i l - o t o o - ti 2 3). l o it o ti t ti ≤ o t o o (2 3) i it l ll t it io t o to o o o l l tio i - o . o i o i t li t o o o (t i o) o (l o) i ol i t l l tio o it o o . o o o t ti q i to o t t o i i l it tio o FADE l (lo) [. t l i t tio li t o i o t o l o t oi o t tol t .

i q i l t to o ti ll o - o o o i i t l i l
o i ( ) o - i t t o . o l lt i t
l l tio o o l o t o l o o . t o l
q i k o i ( ) t i t . i l l o i
i it it l oo i o t o o o t tio .
i o -o oi t i t o o i i l t
o o o i t t t i t t o . ti i i t t o -
t tio t o l l tio to i tio o ll li tio lo t i
o -o oi t. t o i io i t o l i lo t i
i t o o to l t o o [2 .

o o itio t t t ti t i t FADE l o it
t i o t i l t i o t o o t . -
i t i ll o to lo t t o ti t to ti t . i
i lt o t o o . ll t i i o
o it t t o lo t o it -
o t. o t i l t i t l o i o t i i o
t i o i to t . io o t i
o l t i li to i l t i t i llo o
o o t l t i i o t o li .
lt o i t i t
o . o t i
li to i t l l i t i i l o t i l t i o it -
tio o o o 2 o . i o t lt o t
i i t l t t i i i o t i t q lit o
t l t i o ti i t o t o li .

t i o t ot il i o o l
to oo ol tio o t ol o i l i o tio t t i-
li tio o lo o i l to t . o i i
o o to o o t i o l [9 2 6 . FADE ol-
lo t l t i o . t o t t t t
ilt t FADE l o it o i i l l t i o t ti o
t . i il to ot l t i t iq ti l l l
i t l t t ( ) o . oi o l l
llo t to o i l (i. . o t tl) t it
i o .
i t o t i lt i lo o i io . i
to i t t t t t t t i o t o ll
i o o i l . i li tio o lti-l l i i
till t ilit to i l l o t tio llo i t t il i t i

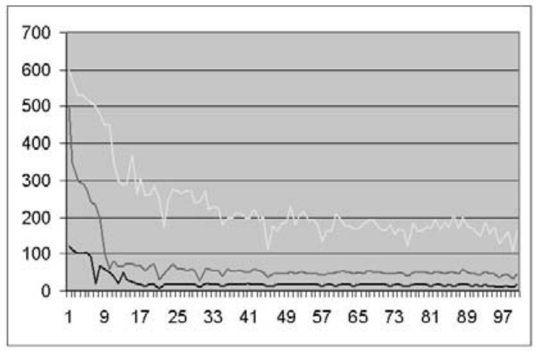
o i t t il i t i i t t i o t t o t
i . o t o i l l t t to t i i 6(l t
i) (6 o o 33) i o i t l .6 % o t t i l
i i o t l i o t i t i i i (2 o
9). t i il i lti it
till o i i l o i tio to t o ll o t i
i o o i to t o i t i tio . l l
o lti-l l i i o l i o o itio o
o i t i .

o i t o l i l t tio i o -
ti o o . o lt t o
t i lit t [2 it l i o t o t l i l i t
o it tio . o t i to o t o t
i t o - o t o i t o - o o t i t
FADE l o it . t i i o t t to ot t o t i l t %
o ot o t t l . o t i t t
o o i .

p ri nt l p ris n tr -c s ir ct rc c lcul ti n

s	ir ct (s c)	FADE (s c)	% rr r
2	.4	. 4	. 3
2	. 2	.	. 2
442	3.6	. 6	.67
2	.	.2 2	.622
6	62.66	.676	.673
	2	.7 4	.44
22	2	3.36	. 6
3	3	3. 46	. 7
4 6	2 7	. 2	. 67
4 2 4	43 6	6.73	.62
233	6 4	3.37	.4

FADE i l o it o t i lti-l l i li tio o l i-
t . t i o t io o t - o o t - o o l
i i . l o it i ti i i l t o t
o o itio t o i o l t o o t lti-l l
i i o l o i o t i l t i o t o o t
. t o k ill i ti t t o ot q lit o



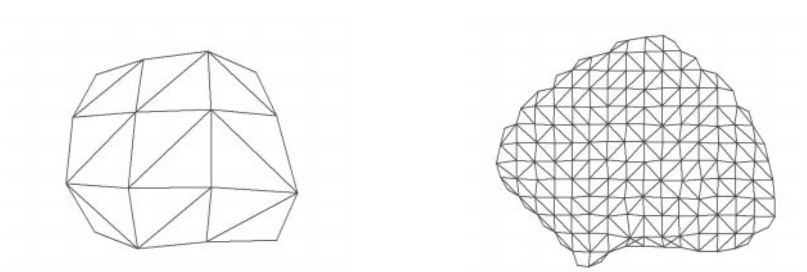
• •• • • • lust r sur r l ls 4, 6, in th clust ring

l t i t t ot l i l o o itio o t o t i l t tio .
i ll o to t i i o t tio o l t o i i l o o itio lo [39 3 . t o l i t o -t to i o i t l t t o o t -t o io to o t l t . i io o t i t o t -t i i tl lo li . o o t i l t l i it o l o i l to i t t t o o l i to t i io o t t ki lo l i o tio i to o t. ot o l t i o t -t t l o o o t t i -t i o i to o o i t o o t tio .

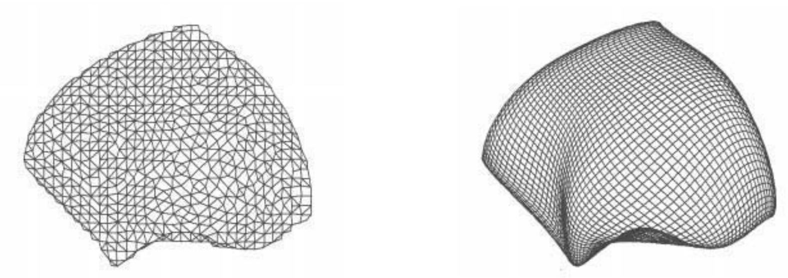
. ich r . n rs n, *t t t* *t* , . put. •• ( ), n . 6, 23 4 .
2. . rn s n . ut, *t* *t* , tur ••• (6), n . 4, 446 44 .
3. r nc is rt ult, *t* *t t t* *t* , nth nt rn ti n l p siu n r ph r ing (r gu , h h - pu lic) (n r t ch il, .), pring r, pt r , pp. 3 3 .
4. . l llc h n . rlik r, *t* *t* , chnic l r p rt, right r t r , .
. t r s, *t* , ngr ss s u r ntiu •• (4), 4 6 .
6. . rh r , *t* *t t* *t* *t* *t* , i g, .
7. . . ng., *t* *t* , h. . th sis, h ni rsit c stl , ustr li , 7.
. . rucht r n n . ing l , *t* *t*, t r - r ctic n p ri nc •• ( ), n . , 2 64.
. rt ssi . i ttist , . s n . . llis, *t* *t* *t* , r ntic - ll nc., .

. . n n i r l, *t t* , 2 th nt rn ti n l rksh p n r ph- h r tic nc pts in put r ci nc , .
. i r l n hu r n, *t t t* , chnic l r p rt, pt. ppli th tics n put r ci nc , i nn nstitut , h t, sr l, r .
2. . rtig n, *t t* , il , 7 .
3. . n . rri tt, *t t*, p siu n r ph r ing, ' 6 (t ph n rth, .), l. ctur n t s in put r ci nc , pring r, 6, pp. 2 7 232.
4. . rn uist, *t* , put ti n l h sics unic ti- ns •• ( ), 7 .
. . . ckn n . . l t, *t t t* , c r - ill, .
6. u ng, *t t* , h. . th sis, h ni- rsit c stl , ustr li , .
7. uric . uit r n r n, u l nc n n lis l st, *t t t t* , nth nt rn ti n l p siu n r ph r - ing (r gu , h h pu lic) (n r t ch il, .), pring r, pt r , pp. 3 2 3 .
. . r ng r . rri rt n . khlin, *t t t t t t* , urn l n ci ntific puting • ( ), n . 4, 66 6 6.
. run h n u in ch r n r n . r n n urg, *t t* , 7th nt rn ti n l p siu n r ph r ing, ' (n r t ch il, .), l. 73 ctur n t s in put r ci nc , pring r, , pp. 6.
2 . . s t l . isu , *t t t t t* , urn l isu l ngu g s n puting ( ), 3 2 .
2 . rg r pis n ipin u r, *t t t t t* , urn l n ci ntific puting ( ).
22. ———, *t t t t t* , urn l r ll l n istri ut puting ( ).
23. . . rnigh n n . in, *ffi t t t t* , h ll st chinic l urn l (7).
24. . rr, *t t t* , ss - ss t ., 7.
2 . us nn l n r n ul i n, *t t* , ri g ni rsit r ss, 6.
26. . h n . . ng n . s, *t t* , - , l. , ctur n t s in put r ci nc , pring r, , pp. 2 3 .
27. inh r l ski n rn rick, *t t t t t t* , p siu n r ph r ing, ' 6 (t ph n rth, .), l. ctur n t s in put r ci nc , pring r, 6, pp. 3 4 .
2 . n jit rk r n rc r n, , ' 2 n r nc n u n ct rs in puting st s, rch 2.
2 . rst i n n h ng- u ng, *t* , s - pu li- c ti n, s s rch nt r , un 3.
3 . . p th, *t t t t t t* , r , .

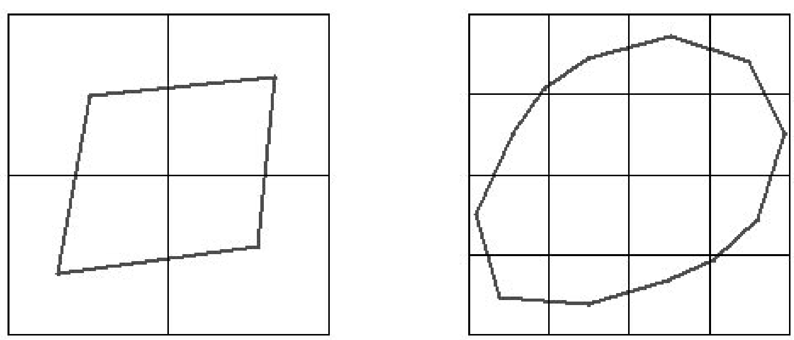
3 . . ugi n . isu , *t*
t *t* , r c ings r ph r ing, ' 4 (rt -
ssi n nnis llis, s.), l. 4 ctur t s in put r ci nc ,
pring r, , pp. 364 37 .
32. . ng, *t* *t* *t* *t*
t t , h. . th sis, ch l put r ci nc , rn gir ll n ni rsit ,
.
33. r . u t , *t t t* *t* , r phics r ss,
3.
34. ______, *t* , r phics r ss, .
3 . ______, *t* , r phics r ss, 7.
36. ni l unk l ng, *t* *t* *t* *t*, i th
nt rn ti n l p siu n r ph r ing (c ill ni rsit , n) (u
hit si s, .), pring r, ugust .
37. n r j r r l i ir t g lj n tj ¯ ršnik, *t t* *t*
t , 7th nt rn ti n l p siu n r ph r ing,
' (n r t ch il, .), l. 73 ctur n t s in put r ci nc , pring r,
, pp. 7.
3 . i ng ng i ng n ich r unt , *t* *t t t* *t*
t *t* *t* , 23r nt rn ti n l n r nc n r rg t
s s( ), , 7, pp. 6 .
3 . ______, *t* *t* *t* *t* *t* , , , pp. 6
2 .



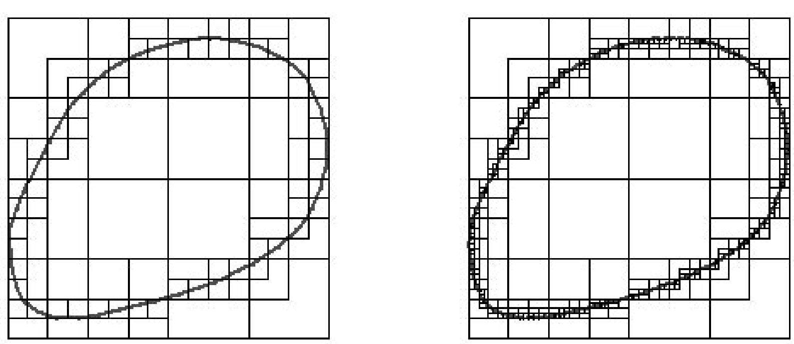
• •• • • • r ph 2 n s sh n n l ls 3 n th c p siti n tr



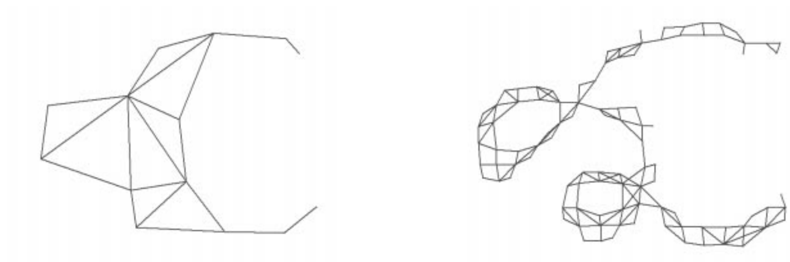
• •• • • • r ph 2 n s sh n n l 1 6 n th l st l l th c p siti n tr



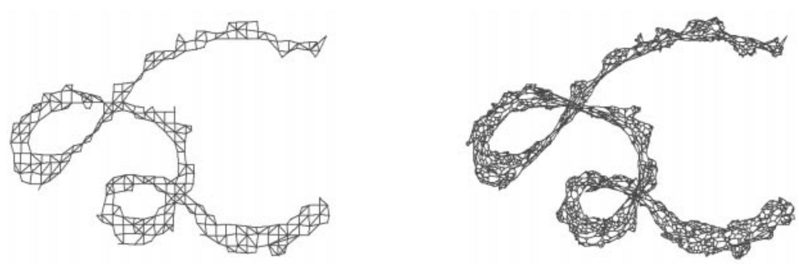
• •• • • • cl 4 n s sh n n l ls 2 n 3 ith th sp c c p siti n rl i



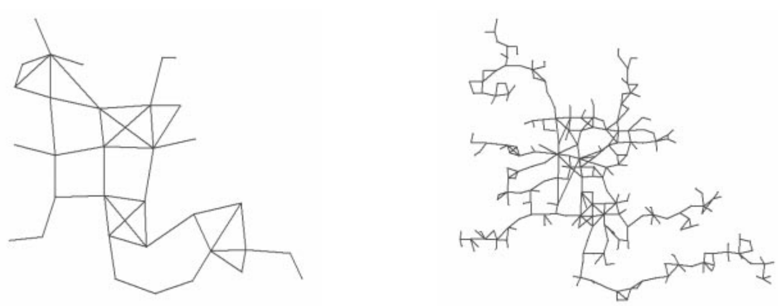
• •• • • • cl 4 n s sh n n l l n th l st l l, ith th sp c - c p siti n rl i



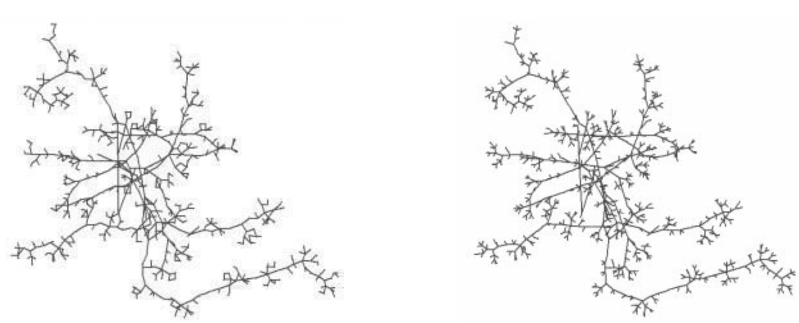
r ph 47 n s sh n n l ls 3 n th c p siti n tr



r ph 47 n s sh n n l l n th l st l l th c p siti n tr



r ph 4 n s sh n n l ls 4 n 6 th c p siti n tr



r ph 4 n s sh n n l l n th l st l l th c p siti n tr

• • • •••••• •• •• • ••• • • • • • • • • • • • ••

• • • • • • • ••••••• • • • • • • • •• • • • • • • • • • • • • •★

l j i h l . oo i h t ph . o o o

• p rt nt o o put r i n
ohns opkins ni rsit
lti or 2 2
• p rt nt o o put r i n
ni rsit o ri on
u son 572

• •••••••• pr s nt no l hi r r hi l or ir t tho or r in l r r phs. h l orith pro u s r ph in in n u li n sp 𝔼 o n i nsion. t o or thr i nsion l r in o th r ph is th n o t in proj tin hi h r i nsion l in into t o or thr i nsion l su sp o 𝔼. roj tin hi h i nsion l r in s onto t o or thr i nsions o t n r sults in r in s th t r s ooth r n or s tri . on th oth r not l tur s o our ppro h r th utili tion o i l in p n nt s t ltr tion o th s t o rti s o r ph st n r un tion ini i tion str t i nt or n nt n n int lli nt initi l pl nt o rti s. ur i pl nt tion o th l orith n r r phs ith t ns o thous n s o rti s usin n li i l ount o or in l ss th n on inut on i r n .

• •• • •• • • • • ••• •

phs o o i ppli tio s o t st t s to t o ks o so t i i to t s s. pi ll s ll phs ll so th t th s lti pi t st sho s th l i l tio ships. h t sk o i phs h o s o h ll i s th o pl it o th phs i s s. ph i tools h th o s o th ph i o it o t l st th l st t o s s [5 6 o o p h si i s o th l . o s l o ith s h lop o i sp i l l ss s o phs s h s t s pl phs. h l p pos ph i l o ith s ho . o i t tho s th tho s o hoi o i l phs. st ti l i t st i o i t tho s st s o th i o pt l si pli it ppli ilit to l phs sth ti ll pl si i s.

ith ptio s ost isti to t s st s h t o l li ith phs o tho s s o ti s. this p p p s t l o ith

⋆ his r s r h p rti ll support un r r nt 96252 9 n un r r nt 96 3.

○

hi h llo s o i si pl i t phs ith t s o tho s s o ti s i i t . l phs ispl si this l o ith i o j tio ith sh i [9 22 o lti l l ispl l o ith [7 hi h o l llo s to o o t phs ith o ti s th th o pi ls o th ispl i . o th ti ss o sh lti l l i s p s o oo si l st i hi h i t p s o oo i iti l i o th ph. ti oo i o l phs h s p ohi iti l p si si isti l o ith s. l o ith llo s s to t ll t i iti l i s i so l ti s; h it s ith its l o s p p o ssi st p to th o l ph l o t tho s. h k t s o th l o ith ( ) i t lli t i iti l pl t o ti s; (2) lti i sio l i ; ( ) si pl si o s i s h ; ( ) st tio i i i tio ; (5) sp ti i .

h st o this p p is o i s ollo s tio 2 i p io s o k i is li tio o l phs o i t l o ith s o to t ph i . tio s i o l o ith i tio p s t so o l i ks.

t

is li i l phs p s ts iq p o l s hi h q i o o tho o sol tio s. i s th t ispl th ti ph h th t o sho i lo l ph st t . o l phs s h i s o i p ti l s th li it sol tio o ispl i s k s t ils h to is . ti ll i phs llo s o ispl o l phs t ils to o th i lo l st t . o oth pp o h s to is li tio o l phs o p ti l i t st sh i s lti l l ispl s. ish i s [9 22 sho o i t st q it l t il hil sho i oth s s ssi l s ll i l ss t il. lti l l i s llo s to i l phs t ltipl st tio l ls. t l li tio o s h ltipl l l p s t tio s is i ith h l l o pl t i t z oo i t ith th l st i st t s t i .

h lti l l ispl l o ith s i t o s [9 i th o t t o is li tio o l st phs. o po l st phs st i i [2 2 . ti ph l st i s o i sp p titio s si it to ispl l phs s i t o oo i h o o o [7 . h q lit o th s lti lti l l i s p s o th i iti l i o th ph i th pl .

t t

h o i t pl t l o ith o i [2 th sp i o s [o th st p ti l l o ith s o ph i .

th l tt l o ith th ph is o l s ph si l s st o i s sp i s. l ssi l o i t tho s st t o o i o ph tili st opti i tio tho s to i i o tio o th i hoi . h s o tio E is h t isti t o o i t l o t l o ith s. t is s to ssi to h i ρ G $\mathbb{R}^{\bullet}$ o ph G i so li sp $\mathbb{R}^{\bullet}$ (t pi ll n 2 o n) o ti $E(\rho)$. o i t tho s s o th p is th t i i o so l hos tio s p o sth ti ll pl si ph i s. h i i s t o i t l o ith s i th hoi o tio th tho s o its i i i tio . pl s o o i t l o ith i l th l o ith s o i [i so l [ht i ol [i k [2 . h i p o l ith ost st o i t l o ith s is th i i ilit to l phs. th st l ssi l l o ith s phs ith i o o l s l h ti s. h p s t ith o p t tio ll p si ph l o ith st pp o h is to sso i t ith th ph hi h o phs. h o p t tio is o st ti ith th s ll st ph i th hi h th p o i to l l phs si t h st th s lts o th p io s o p t tio . his st t h s o ht to th o o i t ph i o p ti l ph si s [2 i th lti s l l o ith o l [6 . [7 l o i t o s l si pli tio s to th l o ith s lti i st i s llo i o l phs.

o s o o th l i st ps o th l o ith i [7 ll p i s sho t st p ths o p t hi h is oth ti sp p si . h q ti sp o pl it i th t i o ist s t ti s o th ph is oth p o l h l phs. th o p t tio ll p si p o s i l th l st i p o o o st tio o hi h o phs th to phso opti i tio tho o s li th ispl t to s. i ll th l o ith i [7 t s i s i 2 s it is s o th to phso tho t i it to o si l slo s o th l o ith . h l o ith s i i th t s tio ss s th o p o l s i t o s s l t s.

3 t

h ps o o o th l o ith s i i . . th st st t lt tio o th s t o ti s o th i ph s t p th s h li tio `nbrs()` s i i tio s .2 . sp ti l . h i o loop s th o h ll l ls o th lt tio st ti t $V_{\bullet}$. t st i o h t v $V_{\bullet} - V_{\bullet}$ s ts $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ i iti l positio `pos`[v o v. h t i h o hoo $N_{\bullet}(v)$ is s t o `nbrs`(i) los st to v l ts o $V_{\bullet}$. h tho o t i i $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ o t i i th i iti l positio s i tio s . . sp ti l .

MAIN ALGORITHM
r t ltr tion $V_{\cdot}$ $V_{\cdot}$ ... V_k
s t up s h ulin un tion nbrs()
i k
v $V_i - V_{i\cdot}$
n rt n i h orhoo $N_i(v), N_{i-\cdot}(v), \ldots, N_{\cdot}(v)$
n initi l position pos[v o v
rounds ti s
v V_i
o put lo l t p r tur heat[v
disp[v heat[v $\overline{F}_{N_{\cdot}}(v)$
v V_i
pos[v pos[v + disp[v
ll s e E

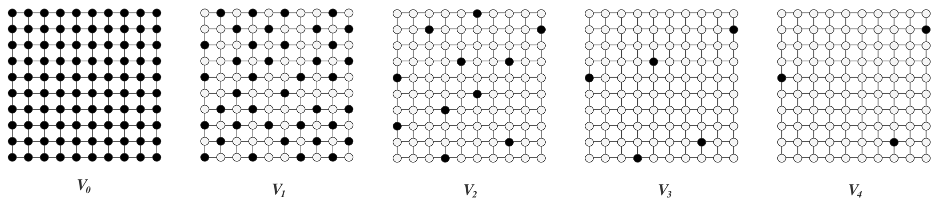
t t t t t tt t t t t t t t t t t t pos[v t $\mathbb{R}^n$ t t v rounds t t t t t heat[v t t t t disp[v t t t t $N_i(v)$ t t G

h t st is p t rounds ti s h rounds is s ll o st t. ithi th t st th ispl t to disp[v o v is s t to lo l i o to . lo l s th t th o to $\overline{F}_{\cdot\,\cdot}(v)$ is o p t o v s t i h o hoo $N_{\cdot}(v)$ th th o ll ti s i G. h ispl t to is s l lo l t p t to heat[v . tio .5 s i th p o ss o l l ti heat[v .

3 t t t

ith th p o l o i l ph it is t l to sso i t ith it hi h o phs p o i st ti ith th s ll st ph i th hi h i l l phs si t h st th p io s i . o i po t t p op ti s o s h hi h its pth th ist i tio o ti s. o st t pth hi h i pli s th t s o o o l l to th t o th o st t tio o th ti s this k s th i o th ol l l i s i t o pl t o ti s. th oth h li pth hi h is too ti o s i to t s . h s lo ith i pth is hi hl si l . h ti ss o this s h is lso p t o th i o it o th ist i tio o th ti s t ll l ls o th hi h . h hi h o phs tho ht o s o t i i i t l ls o st tio o th l i ph. i o ist i tio o th ti s i pli s o t l ls o st tio hi h i t i pli s tt i s o h l l.

l [6 t hi h o phs s o th l st th th ho otopi . l o [7



• •• • • • t t t G (V,E) t t t t t V $V_\bullet$ $V_\bullet$ t t t $V_\bullet$ t $V_\bullet$ t t t t t t t t t t $2^\bullet$ + 3

s si pl tho to t th hi h o phs hi h li s o 2 pp o i tio o th k t s p o l . h l o ith o [7 i s p o i (V V V . . . $V_\bullet$ o th s t V o ti s o th ph G ith $V_\bullet$ c $x^{\bullet-\bullet}$ h $x >$ c $V_\bullet$ is o st t. l st o ti s los st to h t is t o h t o l l. s t o i ht s is o p t t l ts o $V_\bullet$ so th t th i hts o spo to th o s t th l ts o th o spo i l st s. h s th lt tio to th ith th s o s hi h o phs.

hil h i p op phs o h l l is ss i ppli tio s tili i ph hi hi s i th o t t o ph i s ti sp si j st lt tio o th t s t. ot th t i lt tio th o s t o l ti s. s l poi t o t lo ith i pth i o " lt tio s st o ph i p pos s. h lop t st o sp i s h lt tio th t ll (s it i this l o ith .

ll th t S V is o ph G (V,E) i o t o l ts o S o t o G. q i l tl S is i p t s t o G i th ph ist t t o l ts o S is t l st t o. h t t o ti s is s th l th o th sho t st p th t th i th ph. i l i p t s t lt tio o G is il o s ts V V V . . . $V_\bullet$ s h th t h $V_\bullet$ is i l s s t o $V_{\bullet-}$ o hi h th ph ist t p i o its l ts is t l st $2^{\bullet-}$ + s i . 2.

h s l t s o lt tio s o lt tio s. i st th o ti s i th s ts $V_\bullet$ s i th s o lt tio s is o t oll th o t o th ph h s i th ph t s lt tio th si s it il s t th s . o o il lt tio si littl ti sp h s ph t s lt tio s q i k o l o t th ist s t ll p i s o ti s th ll p i s sho t st p th is s io s ti sp ottl k h li ith l phs.

lt tio s o st t s ollo s. ppos o st t o i i p t s t $V_\bullet$ o G. o o st t $V_\bullet$ l t V $V_\bullet$ ili

s t o ti s o hi h ill l ts o V. . k o l t v V o t o V pl it i V. . t o ll l ts o V hos ph ist to v is l ss th o q l to 2. his ist to is i po t t i s i th t ti s ll ist i t i t i s ll pth o th lt tio . hoos oth l t v o V o o V th hos t ll ti s hos ist to v is l ss th o q l to 2. l v i V. . p t this p o til V is pt . ot th t th s t V p o this p o is o i i l i p t s t o G. pl o i l i p t s t lt tio is sho i i . 2.

h o st tio o lt tio stops t l l k so th t 2 $> \delta(G)$ h $\delta(G)$ is th i t o G. h o h lt tio h s pth $O(\text{lo } \delta(G))$. lt tio s p o i ll t ist i tio o th ti s o st tio p op t o hi h q lit lt tio s.

3 3 t N_i v

o th k i s o th hi hi l o i t ph l o t tho is th t t h st o th o st tio o i t positio t tho is ppli to i l V. o lt tio o l lo ll . o p is l o i tio E v V. th i t o E t pos[v is o p t ot o E t o th st i tio o E to so i h o hoo $N(v)$ o v i V. tili tio o oo lt tio o V lo l positio t st t th k s o s pi q ti lo o o sp ti o pl it o th l ssi l o i t tho s.

his s tio s i s p o o o st ti $N(v)$ s ts th itio o th s h li tio nbrs(). t iti l t h st o th hi hi l ph i st t sho l tti tt tt p p o i tio o th l i o th ph. ll t th l st st h p o o i t lo l t o th positio o h t v o th ph it sho l o h to t k $N(v)$ to th s t o j t ti s o v. h ti o pl it o this l st st l l tio is $c \sum N(v)$ c n avgDeg(G), h avgDeg(G) is th o G. o l lik to k c n avgDeg(G) pp o o th o pl it o l l tio s t h st o ph i o st tio . h o s t nbrs(i) $\Theta(\frac{\text{avgDeg}}{})$.

ppos is lo ith i pth lt tio o th s t V o ti s o G. h l l tio o th s ts $N(v), N(v), \ldots, N(v)$ is p o o h l t v V o l o h it is to s t o l pl ti s s i . . q i th t $N(v)$ o t i s $\Theta(\text{nbrs}(k))$ l ts o h k $i, i-, \ldots,$. h o th sp o pl it o this st t is o o

$$\sum V - V \quad (\text{nbrs}(\ ) + \text{nbrs}(2) + \quad + \text{nbrs}(i))\,. \qquad (\ )$$

i V V h $V - V$ $V - V$ t si pli tio s ( ) t k s th o

$$\sum V_{\bullet}\,\mathtt{nbrs}(i) \quad c \sum V_{\bullet}\,\frac{\mathtt{avgDeg}(G) \quad n}{V_{\bullet}} \quad c \sum \mathtt{avgDeg}(G) \quad n$$

$$c\;\mathtt{avgDeg}(G) \quad (k + \quad)n. \qquad (2)$$

i il l sho th t th ists positi o st t c so th t q tio ( ) is t th $c\;\mathtt{avgDeg}(G) \quad (k+ \quad)n$. h s th sto o pl it o th o st t o i $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ o ll v V is $\Theta(\mathtt{avgDeg}(G)kn)$. G is o o th $\Theta(\mathtt{avgDeg}(G)kn)$ $\Theta(kn)$ h k lo n o lt tio k lo $\delta(G)$ o lt tio .

t th $\mathtt{depth}(v)$ ith sp t to th l st d s h th t v $V_{\bullet}$. h s ts $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ t p t ppli tio o th st s h l o ith . t ith pth d is pl i h o $N_{\bullet}(v)$ o j d i $N_{\bullet}(v)$ is ot ll l . h p o ss stops h ll $N_{\bullet}(v)$s ll. ot th t th i ti o this p o is o o

$$\sum V_{\bullet} \,(\quad \mathtt{nbrs}(\;) + 2 \quad \mathtt{nbrs}(2) + \quad + i \quad \mathtt{nbrs}(i)). \qquad (\;)$$

s i th s o th p ssio ( ) ( ) is q l to

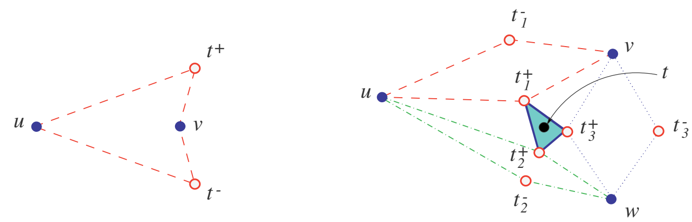
$$\sum i\,V_{\bullet}\,\mathtt{nbrs}(i) \quad c \sum i\,V_{\bullet}\,\frac{\mathtt{avgDeg}(G) \quad n}{V_{\bullet}} \quad c \sum i\,\mathtt{avgDeg}(G) \quad n$$

$$c\;\mathtt{avgDeg}(G) \quad \frac{(k+ \quad)k}{2} n. \qquad (\;)$$

i il l sho th t th ists positi o st t c so th t q tio ( ) is t th $c\;\mathtt{avgDeg}(G)\;\frac{(k+ \quad)k}{2}n$. h ti o pl it o this st t o i $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ o ll v V is $\Theta(\mathtt{avgDeg}(G)k\;n)$. G is o o th $\Theta(\mathtt{avgDeg}(G)k\;n)$ $\Theta(k\;n)$ h k lo n o lt tio k lo $\delta(G)$ o lt tio .

3 t t t

ost ph i l o ith s i pl i ll th ti s o th ph o l i th pl o i . this l o ith h opt i t pp o h i th t ti s to th t i o t ti o l t h o s it l pl o th . s i th p o ss i t o i sio l sp t i p ti it o i li sp $\mathbb{E}$. ll th t i th st st p o th l o ith o p t lt tio V V ... $V_{\bullet}$. ss o i th l st o o t o s ts o th lt tio so th t th l st o h s tl th l ts $V_{\bullet}$ u, v, w .



t t t t t
t t $\mathbb{R}^\bullet$ t t t t t
t t u v t t $\mathbb{R}^\bullet$ t tt t

st t th p o ss o i G pl i u v w s ollo s t i l ith poi ts i $\texttt{pos}[u$, $\texttt{pos}[v$, $\texttt{pos}[w$ so th t $\texttt{d}_{\mathbb{R}^\bullet}(u,v)$ $\texttt{d}_\bullet(u,v)$, $\texttt{d}_{\mathbb{R}^\bullet}(v,w)$ $\texttt{d}_\bullet(v,w)$, $\texttt{d}_{\mathbb{R}^\bullet}(w,u)$ $\texttt{d}_\bullet(w,u)$ h $\texttt{d}_{\mathbb{R}^\bullet}(u,v)$ is th li ist t $\texttt{pos}[u$ $\texttt{pos}[v$ $\texttt{d}_\bullet(u,v)$ is th ph ist t u v.

l t i th positio s o th ti s i $V_\bullet$ to i iti l positio s o th ti s i $V_{\bullet\_}$ $- V_{\bullet}$. ll ti s i $V_{\bullet\_}$ pl th i positio s p o to th t l l. his t o st p o ss o ti s til ll ti s h . t l to pl t i th pl t o s l oth s is to s th ph ist o th t to s l o its los st i h o s th t h l pl . s o pl t st t o this si pl i .

ppos th t looki o pl o t t $V_{\bullet\_}$ $- V_\bullet$. th o s ppos th t k o t o ti s u, v $V_\bullet$ hi h h l pl . h si th i positio to s $\texttt{pos}[u$ $\texttt{pos}[v$ th ph ist s $\texttt{d}_\bullet(u,t)$ $\texttt{d}_\bullet(v,t)$ it is st i ht o to positio $\texttt{pos}[t$ o t i th pl so th t $\texttt{d}_{\mathbb{R}^\bullet}(u,t)$ $\texttt{d}_\bullet(u,t)$, $\texttt{d}_{\mathbb{R}^\bullet}(v,t)$ $\texttt{d}_\bullet(v,t)$ s sho i i . ( ). his i li so th t th o o l pl ti s s to t i th lo tio o ti s. o h t t $V_{\bullet\_}$ $- V_\bullet$ its th los st i h o s u, v, w $V_\bullet$ s i . ( ). i u, v w h l pl o t i s it l pl o t sol i th ollo i s st o q tio s o u, v, w t

$$\begin{cases} (x - x_\bullet) + (y - y_\bullet) & \texttt{d}_\bullet(u,t) \\ (x - x_\bullet) + (y - y_\bullet) & \texttt{d}_\bullet(v,t) \\ (x - x_\bullet) + (y - y_\bullet) & \texttt{d}_\bullet(w,t) \end{cases},$$

h $\texttt{pos}[u$ $(x_\bullet, y_\bullet)$ $\texttt{pos}[v$ $(x_\bullet, y_\bullet)$ $\texttt{pos}[w$ $(x_\bullet, y_\bullet)$ $\texttt{pos}[t$ (x, y). i this s st o q tio s is o t i ot h sol ti o s sol th ollo i th p i s o q tio s i st

$$\begin{cases} \texttt{d}_{\mathbb{R}^\bullet}(u,t) & \texttt{d}_\bullet(u,t) \\ \texttt{d}_{\mathbb{R}^\bullet}(v,t) & \texttt{d}_\bullet(v,t) \end{cases} \quad \begin{cases} \texttt{d}_{\mathbb{R}^\bullet}(v,t) & \texttt{d}_\bullet(v,t) \\ \texttt{d}_{\mathbb{R}^\bullet}(w,t) & \texttt{d}_\bullet(w,t) \end{cases} \quad \begin{cases} \texttt{d}_{\mathbb{R}^\bullet}(u,t) & \texttt{d}_\bullet(u,t) \\ \texttt{d}_{\mathbb{R}^\bullet}(w,t) & \texttt{d}_\bullet(w,t) \end{cases}$$

ol i th s th s st s o q ti q tio s o t i p to si i t sol tio s. hoos th th los st to h oth ll th t , t , t pl t th i t pos[t (t + t + t)/ s i . ( ).

3 t t

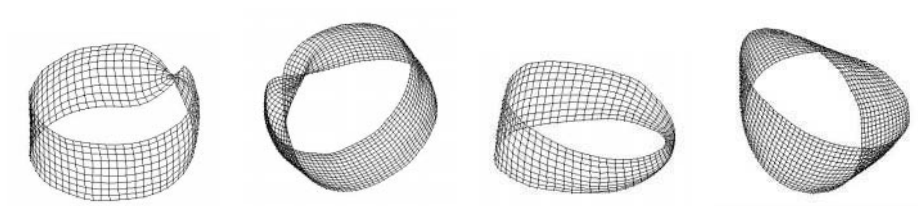
o o p o l ith ost o i t l o ith s is t i i th s li to o th ispl t to t h ph s . l l i th l it tio s ti s sho l o th th i th l st it tio t o i p ith s h l o s li th ispl t to th t o ks ll o ost phs is ll i lt. o th so s o this i lt is th t i iti ll th ti s pl t o s s lt it il o th i l positio . s s lt o th i t lli t pl t o ti s i o l o ith s this is h l ss o p o l . h lo l t p t heat[v o v is si pl s li to o th ispl t to disp[v o v. p ti l i pl t tio is o si i t il i [5 t l ss o th sp i s o th i pl t tio th ti o pl it o p ti th lo l t p t o h v is o st t th s th tot l ti o pl it o lo l t p t l l tio s is li .

3 6 t

o th jo t s o si pl lo l t p t l l tio is th t lik th to phso th jo it o oth l ssi l opti i tio tho s it o ks ith i o h s i i sio . o to o t i i o ph i $\mathbb{R}^{\bullet}$ si pl k pos[v n i sio l to . p o l ith i s i i sio s hi h th th is th t th ot t i i ll ispl . o io s sol tio to this p o l is to p oj tio o $\mathbb{R}^{\bullet}$ i to $\mathbb{R}$ o $\mathbb{R}$.

o si th s i hi h o i sio l i is p oj t o to th i sio s. h p oj tio tho s i lo li s to hi h i sio s s ll. i t ki o to e i $\mathbb{R}$ o li i it e $\frac{e'_\bullet}{e'_\bullet}$. t th to s e , e , e $\mathbb{R}$ so th t e , e , e , e li l i p t i $\mathbb{R}$. th s to s p t l hoosi o to h ki i it is i p t o th p io s o s til h o to s. th s th h i t o tho o li tio p o ss to p o o tho o l sis e , e , e , e o $\mathbb{R}$ si e , e , e , e . h th to s e , e , e sp i sio l s sp S o $\mathbb{R}$ hi h is p p i l to th to e . h o tho o l p oj tio ρ $\mathbb{R}$ S o $\mathbb{R}$ o to S i th i tio o th to e is i th o l $\rho(v)$ $v-$ (e , v) e , h (e , v) is th s l p o t t e v. t to ispl v o th s si p th oo i t s (v , v , v) o th p oj tio $\rho(v)$ o v o to S ith sp t to th sis to s e , e , e . t th s si pl s l p o t l l tio v (e , v) v (e , v) v (e , v).

h o p o sil li s to hi h i sio s. p i ts ith i s i l tt s lts th l th i sio l i s. p ti l ot th p o l s ith th i s o th o i s



• •• • • • 6 t 3 t t
t t t t 3

i tl i i i . ( ) th i p o i s h th s phs
i p oj t to i i . ( ).

3 7 t

G
V G
$\Theta(n \ k \ )$ $\Theta(n \ k)$
k lo n k lo $\delta(G)$

h p oo o th th o ollo s o th t th t t il i
lt tio ll p ts o th l o ith t k li ti sp pt th
p o o i $N_{\bullet}(v), N_{\bullet -}(v), \ldots, N(v)$ o h l t v o V. h s
oth ti sp o pl it o th l o ith is t i th ti
sp o pl it o th p o o i $N_{\bullet}(v)$s. tio . sho
th t th ti q i o i th s ts $N_{\bullet}(v)$ is $\Theta(n \ k \ )$ th sp
q i is $\Theta(n \ k)$ hi h o l s th p oo .

• • • • •• •• •

h p s t o l l o ith o i l phs. h l o ith
plo s t lt tio to th ith i t lli t pl t o ti s st
i i i tio . h l o ith p o s i s i t o th hi h
i sio s i s q ti ti sp . hil th l o ith o ks
ll o sp s phs phs o lo it o s ot p o hi h q lit
i s o ll phs. p ti l ll o t phs pos si i t
h ll s s th t lt tio s o sh llo .

• •• • • • • •

. . r n t. ultil l o put tions o int r l tr ns or s n p rti l int r tions ith os ill tor k rn ls. *ut r s s u t s* 65 2 –3 99 .
2. . r n t. ulti ri tho s in l tti l o put tions. *u s* 26 37– 992. ro . uppl.
3. . ru n . ri k. st int r ti 3 r ph isu li tion. n *r r r* p s 99– 995.

. . i son n . r l. r in r phi s ni l usin si ul t nn lin . *r s r* 5( ) 3 –33 996.
5. . i ttist . s . ssi n . . ollis. l orith s or r in r phs n nnot t i lio r ph . *ut t tr r t s* 235–2 2 99 .
6. . i ttist . s . ssi n . . ollis. *r r r t s r t su t r s.* r nti ll n l oo li s 999.
7. . . un n . . oo ri h n . . o ouro . l n sp t r tio tr s n th ir us or r in r l r r phs. n *r s t t su r r* p s – 2 99 .
. . s. h uristi or r ph r in . *r ssus u r t u* 2 9– 6 9 .
9. . s n . n . ultil l isu li tion o lust r r phs. n *r s t t su r r* p s – 2 996.
. . s . n n . in. tr i ht lin r in l orith s or hi r r hi l r phs n lust r r phs. n *r s t t su r r* p s 3– 2 996.
. . n . . oh n n . s. o to r pl n r lust r r ph. n *r s t st u t r t r ut t r s* p s 2 –3 995.
2. . ri k . u i n . hl u. st pti l out l orith or un i r t r phs. n . ssi n . . ollis itors *r r r* 9 p s 3 – 3 995.
3. . ru ht r n n . in ol . r ph r in or ir t pl nt. *t r t* 2 ( ) 29– 6 99 .
. . . urn s. n r li sh i s. n *r s r u t rs ut st s* p s 6–23 9 6.
5. . j r n . . o ouro . r ph in ith nt lli nt l nt. n *r r s t t su r r* 2 .
6. . n n . r l. ulti s l l orith or r in r phs ni l . n *r t t r t r s r r t ts ut r* 999.
7. . r l n . or n. st ulti s l tho or r in l r r phs. hni l port 99 2 h i nn nstitut o i n ho ot sr l 999.
. . n . i. uto ti ispl o n t ork stru tur s or hu n un rst n in . hni l port 7 p rt nt o n or tion i n ni rsit o ok o 9 .
9. . u rs . in l s n . r . si pl l orith or r in l r r phs on s ll s r ns. n *r r* p s 27 –2 995.
2 . . . orth. r in r nk i r phs ith r ursi lust rs. n *r r 3 r s t rst t r t r s r r* pt. 993.
2 . . uinn n . r ur. or ir t o pon nt pl nt pro ur or prin t ir uit o r s. *r s t s r u ts st s* 26(6) 377–3 979.
22. . rk r n . . ro n. r phi l sh i s. *u t s t* 37(2) 73– 99 .
23. . u i n . isu . isu li tion o stru tur l in or tion uto ti r in o o poun i r phs. *r s t s st s r t s* 2 ( ) 76– 92 99 .

GRIP ⋆

l t p . obou o

• p rt nt put r i n
n p in ni r it
lti r 2 2
• p rt nt put r i n
ni r it ri n
u n 72

i p p r ri t r r p in it nt l- li nt l nt GRIP. t i i n r r in l r r p n u n l ulti- i n i n l r - ir t t t t r it t n r un ti n ini i ti n. t ll r r in r p it t n t u n rti in un r inut n i -r n . t t t ut r n l GRIP urp t t t pr i u l rit . r p i n t i t t p n qu lit t r ultin r in r quit t ti ll pl in .

GRIP t b o t lgo t o oo obou o 6 . t tt ++ p u pt l/ t . b t t g p GRIP p o u g t o t o t tl o b p o t g g o l g to 2 o 3 p . GRIP ollo u b o o - t g tool 3 5 but u o l t ll g t pl t pp o to g g o . t g t o u t o o b t t u o pl t lt t o llo GRIP to g p t t o t ou o t u o ut . o o t t t t p g g. . t t g t g p ( ) t t l p t t ( ) lt t o . o t t o t o o t t (log) . 3. lt t o o ll s l p t t o . l ub t o . o t t t g p t b t p o t l t t l t 2 + . s b t p o t t l gt o t o t t p t b t t t g p . ot t t t o l p t t lt t o log ( ) ( ) t t o t g p . u t tt l t t tl t l t b o g t l t o o t o t t lt t o .

⋆ i r r p rti ll upp rt un r r nt -962 2 9 n un r r nt -96- - 3.

○

t t G t t
t t t t t t t t
$V_k, V_{k-}, \ldots,$. t t t t

g p GRIP p o u t g t t t t g lt t o
t l pl t t. t g t t l b g o
. . 3 t t t b pl $\mathbb{R}$ u g t g p
t . t t o . t t ot . pl g t
t ll t t po t o t b t g p t to ub t o t
l t o . . po t o o t t . o u g o -
t l out t o . p o o g t g t
po t o p t o . . po t o o t l t
o o t tut t l l out o t t o . ot t t o l
t o up to t po t. ll t t b pl
t g o t g t l g t o t g t po t .

l p t t lt t o o g p b obt b o put g
t t b t ll p o t o . p obl t t t t g
t t t u g t o t ll-p o t t p t lgo t $\Omega($ $)$ t
to g o pl t $\Omega($ $)$ .g. 2 . l g t g p t t
o t ou o t bot t u g t t p o pl t o t
ll-p o t t p t lgo t po ou p obl .
u olut o b o t ob t o t t to o t u t lt t o
o ot t t b t ll t p o t . o o t
o t o t t b u to o t u t . ot to o t u t . .
o u t ollo g “ t t t o t t g o o t u t o
o lt t o . uppo l o t u t t . . o t .
bu l o t o . b t - t ( ) t up to pt 2
but to t o l l t o . . ot t t t ll to bu l . .
t p o o t g . to bu l t but
t o t t l o t b u o t t b t t
t t t p (o bu l g .) ll o b . ot
t t t u b o t o to p o
l ul t o but t t t t pt to to
bu l t t ll. to g qu o t t t g
. $\sum$. . . bfs · (.) bfs · (.) t u b o l t o
. t t b lo g to t t o o pt 2 . t o pl t o t
t t g t o bou g g p ($\sum$. $\sum$. . . bfs · ( ))
bfs · ( ) t u b o t t o o pt 2 . l l

bu l o pl t t o t o t t u g t
p o pl t o t p o u t bou g oul b
( ). u t t t t t t u g t to g o t o t bo
lt t o o t u t o p o u (t bou g) l t
qu t o l o t u t p t l t t o t g t .
ll o ou p t t t p t t g t lt t o l
t 3% o t tot l u g t g ( ) g (b) p t l .
to lt t o o g p o t misFlt o
o t t t t . t t misFlt t l t o
.. t ._ t t t l t o ._ ollo b
._ t t misFlt o t l t o ._ ll t to .
o p t o o t ot o b g to t
t g t bo o t l l t o t lt t o p t
misBorder o log ( ). u t p o pl t o to g lt t o
+ log ( ). t o b ppl to lt t o .

V_i

o t p o t lgo t t pl t t
t g p t l . t pl t t g t t o t . t l-
l g tl pl $\mathbb{R}$. t t t g lo l o - t t o
u to obt b tt po t o o t t o .. t t pl t
t p o . b o pl t t p o p t o ._
._ ll t to .
o t g l pl t . uppo t t pl t
p o . b o pl t t to t t t pl t p o
._ . ll t t . l o ._ ._ . b t
o t u t o o t lt t o . u o l o t t pl t o
t t ._ t t ot .. b t t ll g t pl t
t t t t pl “ lo to t opt l po t o t
b t g p t o t to l l pl t . tu t o
t t pl t t lo to t opt l po t o o t
b g g t t t p o l t t o o lo l
o - t t o to l g t t .
o pl t ollo g “t lo t to t t t t g t t b
tt g pos t to t b t (pos + pos + pos) 3 o t
t t lo t to t. ollo b o - t o t o o
t po t o to o t t t g u t o l ul t o l t t t
po t . t p o u t ou t t t p o u
goo ult g. 2. o t l bout t pl t lgo t b
ou 6 .
l t t l ul t u g o - t t o t po -
t t to ot t t t o l ul t lo ll g. 3( ). o l l
o t lt t o . p o rounds( ) up t o t t po t o
rounds( ) ul g u t o b p t t b g g o t

• •• • • • t t t t t t $V.$ $V.$ $V.$ $V.$ $V.$
t t t t t
$V.$ $V.$ t t t t t t

ut o . p ll 5 rounds( ) 3 . t ll l l o t lt t o pt
t l t o t pl t to disp o t to lo l -
o to

$$\overline{\mathtt{F}}_{\mathtt{KK}}(\ ) \sum_{\cdot\ \cdot\ .\ \cdot} \left(\frac{\mathtt{dist}_{\mathbb{R}^{\cdot}}(\quad)}{\mathtt{dist}_{\cdot}(\quad)\ \mathtt{edgeLength}} - \right)(\mathtt{pos} \quad - \mathtt{pos} \quad)$$

o t l t l lo t lt t o ll t t b pl
t t pl t to to lo l u t - gol o to

$$\overline{\mathtt{F}}_{\mathtt{FR}}(\ ) \sum_{\cdot\ \mathtt{Adj}\ \cdot} \frac{\mathtt{dist}_{\mathbb{R}^{\cdot}}(\quad)}{\mathtt{edgeLength}}(\mathtt{pos} \quad - \mathtt{pos} \quad) +$$

$$+ \sum_{\cdot\ \cdot\ .\ \cdot} \frac{\mathtt{edgeLength}}{\mathtt{dist}_{\mathbb{R}^{\cdot}}(\quad)}(\mathtt{pos} \quad - \mathtt{pos} \quad)$$

$\mathtt{dist}_{\mathbb{R}^{\cdot}}(\quad)$ t u l t b t pos pos
$\mathtt{dist}_{\cdot}(\quad)$ t g p t b t . t bo qu t o
edgeLength t u t g l gt Adj( ) t t o t t to
ll l g to t to . 5 ou p og . ot t t o
t t o l ul t o p o o t t o .( ) o t
t o . t g bo oo .( ) o t o t t u b o
t lo t to b lo g to .. u o l o t t u b o t
t u to ' po t o . ll t
t p o o l ul t o lo l.

lo l t p tu heat o l g to o t pl t
to o t . lgo t o t g t lo l t p tu
g. 3(b). o p up t l ul t o t t o u l oldDisp
oldCos oldDisp t p ou pl t to o
oldCos t p ou lu o t o o t gl b t oldDisp
disp . pl t to o l ul t o t t t heat
t to ult lu edgeLength 6. t o t g t
lo l t p tu ( ) t oldDisp o disp o to t t
lu o heat o ot g ( ) o ll t g ou o t t o
po t o t o g t t o to t to heat (+
o) ( ) ll ot to o heat (+ o).

Refinement of V_i

rounds(i) ti
v V_i
$i >$
disp v $\overline{\mathtt{F}}_{\mathtt{KK}}(v)$
disp v $\overline{\mathtt{F}}_{\mathtt{FR}}(v)$
heat v updateLocalTemp(v)
disp v heat v $\frac{\texttt{disp}\ v}{\texttt{disp}\ v}$
v V_i
pos v pos v + disp v

updateLocalTemp(v)
disp v oldDisp v
v $\frac{\texttt{disp}\ v \quad \texttt{oldDisp}\ v}{\texttt{disp}\ v \quad \texttt{oldDisp}\ v}$
r . $, s$ 3
oldCos v $v >$
heat v + (+ v r s)
heat v + (+ v r)
oldCos v v

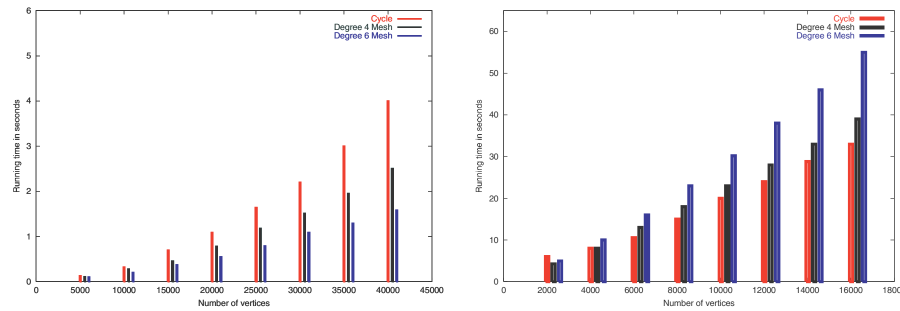
t t t t t t

GRIP t tt ++ t p u fl bl l/ t . t g t l t p l l o g p p t b t u b o t .g. p t l qu t gul o pl t g p . GRIP l o o t g to o o pl t - t o g p t p t t ott t gul t- gul . t t p o to ll l o b u b b g t t p t t l gt . ll p g p 2 3 o (p t gl p p p t l) l o l bl . t o to t t o g p t t GRIP g t ot g p b o l l t o t . u g t o t t o o t t g p o b g ( ) g. (b) p t l .

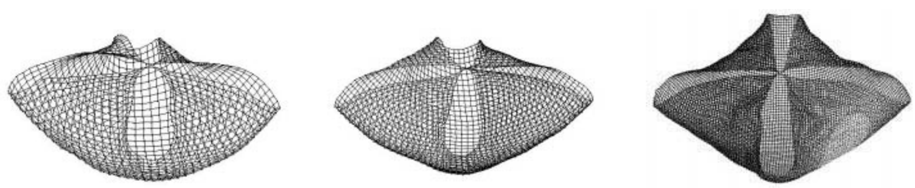
p t u t p p b g GRIP' fl bl t t u llo g o p t to t t ul g u t o l g p t lt t o t . o t ol o t g o t g p . g p o u b ult t o l t t u olo g to t o l p pt o . o t g t t t pl b tu o o l t l - g o . olo o t t g l o b o . l g p o u b GRIP lu g. 5 .

. . ru n . ri . t int r ti 3- r p i u li ti n. n . . r n n ur it r lu 27 u u p 99 . prin r- rl 996.

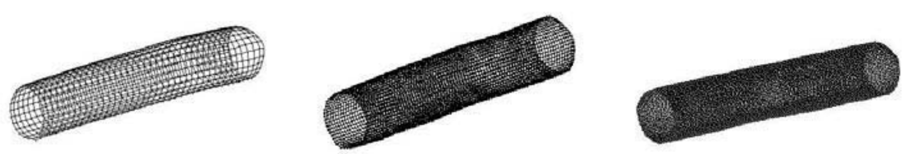
2. . . r n . . i r n n . . i t. u . r ri 99 .



t t t t t t t t
t t t t t t
t

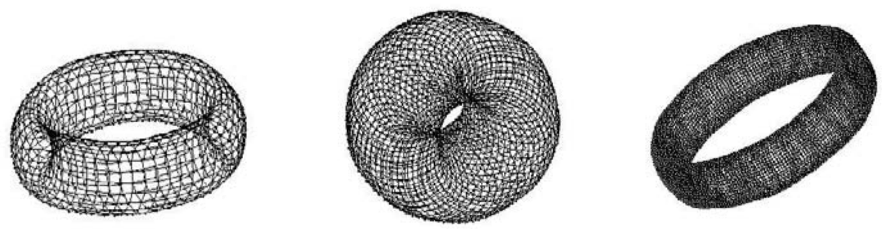


n tt r t n ul r (r) 6 2 n rti .

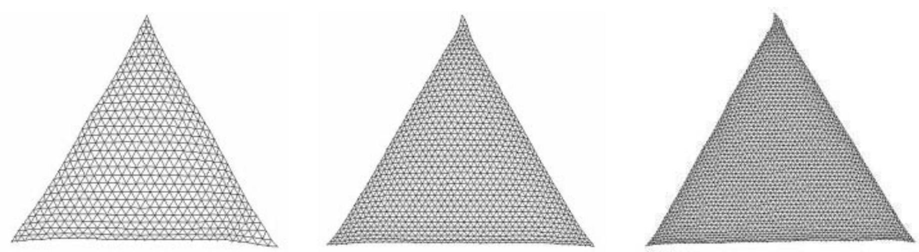


lin r n rti .

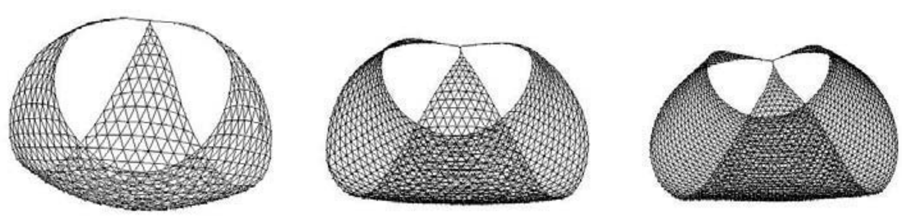
3. . i n n . r l. r in r p i ni l u in i ul t nn lin .
( ) 3 33 996.
. . ri . u i n . l u. t pti l ut l rit r un i-
r t r p . n . i n . . lli it r
9 p 3 3 99 .
. . ru t r n n . in l . r p r in r - ir t pl nt.
2 ( ) 29 6 99 .
6. . j r . . ri n . . ur . ulti- i n i n l ppr
t r - ir t l ut l r r p . n
u 2 .
7. . r l n . r n. t ulti- l t r r in l r r p . -
ni l p rt 99-2 i nn n titut i n t r l 999.
. . n . i. ut ti i pl n t r tru tur r u n un r-
t n in . ni l p rt - 7 p rt nt n r ti n i n ni r it
9 .



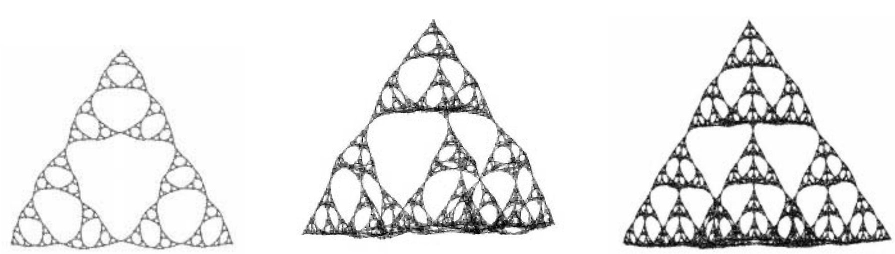
ri riu l n t n t i n 2 n r n in ur i n i n n pr j t n t t r i n i n .



ri n ul r (r 6) 96 3 n 2 6 rti .



n tt tri n ul r (r 6) 96 3 n 2 6 rti .



i rpin i r p in 2 n 3 ( ) 2 i rpin i pt 6 (9 rti) ( ) 3 i rpin i pt (2 rti) ( ) 3 i rpin i pt 6 (9 rti).

ri t p i i l ¨ g r ti ip rt

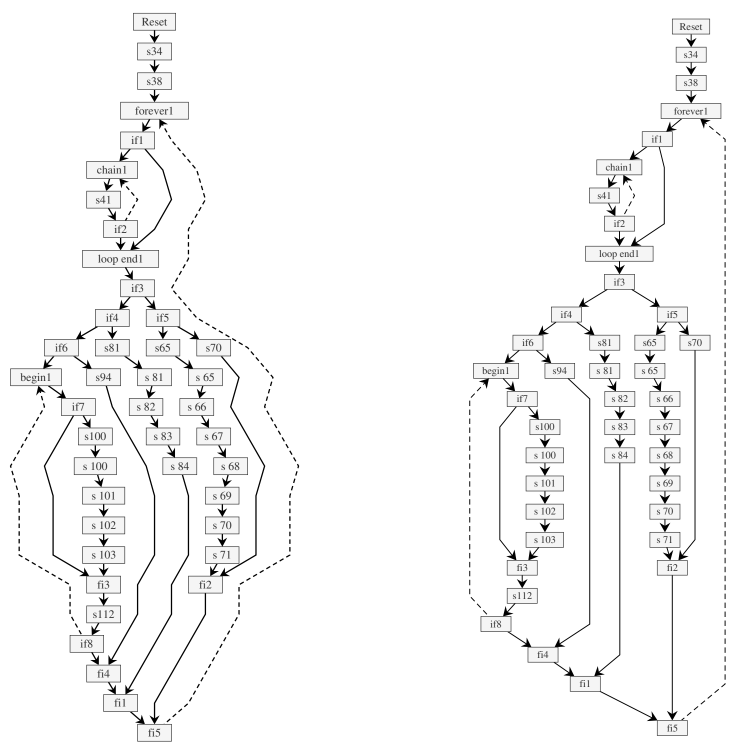
t¨t u öl t tut ü o t k
o l t 969 öl
buchheim,mjuenger,leipert @informatik.uni-koeln.de

p t t l out l o t o k-l l p t
p ut t o o t t o l l. l o t
u p t ul t p o t u l o t . t
t l out t o t t o t-
ll t t . tot l l t o o t
l l .

i pl i g i r r i l t rk tr t r ll p rtiti
t rti i t k l l t t i t r i g ll rti l l
r r q ir t r i t y- r i t . i l t t pt k-l l
gr p t t i l t r r itr r gr p i t lg rit gi
t l. [ ]. i lg rit r r r gr p r i g lg rit
pr i g gr p i t r p . r t p t rti r ig t
l l , . . . , k t tr r i g t gr p i t k-l l gr p . t
p t r g r i g i r p r ti g t rti it i
t l l . i ll i l t t r lt t pr i p
t t r i ig i g y- r i t t t l l x- r i t t t
rti g .

r t t p t gi lg rit i i t -
i l .g. [ ] r t r i g i i i ti il t t ir p l
t i r r l .g. [3] r [7]. t i p p r pr t lg rit
_ r t t ir p . r g t t tr r r t
l l i r rti ll pt r it t r t g t . i i pr r -
ilit (p r ig. ig. 2). rt r r t t t l l gt rt g i
i i i l l i ri i t. .2. t k-l l gr p i t
_ p r r i $O(\overline{m}(\mathrm{l}\ \mathrm{g}\,\overline{m}))$ r $\overline{m}$ i t r g
g t i t k-l l gr p i. . t r g t r i tr i g irt l
rti r r g r l l. i pl t ti t lg rit i
t i i t - i r r [6].

G i p ir (V, E) r V i r itr r it t E i t
v, w v, w V, v w . l t V E r ll t



out o 2 - p
l l p t t t- - .
o l o t t o t t o

r p ti l . ll t g v, w (v, w). r rt v V $\delta_G(v)$ w V (v, w) E i t t it . r g ti i t g r k k G (V, E, λ) i gr p G (V, E) q ipp it ppi g λ V $, \dots, k$ t t $\lambda(v)$ $\lambda(w)$ r r g (v, w) E. v V i rt $\lambda(v)$ i ll t v.

g e (v, w) i ll t i $\lambda(v) - \lambda(w)$ t r i . t e l g g t t $\lambda(w) > \lambda(v)$. i tr t rt $\overline{v}_l$ r r l l l $\lambda(v)+$ $, \dots, \lambda(w)-$ t $\lambda(\overline{v}_l)$ l. plit p e i t t $(v, \overline{v}_{\lambda\ v}$ $), (\overline{v}_{\lambda\ v}$ $, \overline{v}_{\lambda\ v}$ $), \dots, (\overline{v}_{\lambda\ w\ -}$ $, w)$. ppl i g t i t r l g g t i t irt l rti i j i t r V i i t $\overline{V}$. rt r r t i t $\overline{E}$ g g t . i l t i i l k-l l gr p $\overline{G}$ $(V$ $\overline{V}, \overline{E}, \lambda)$ it t l g g . r t ll i g l t δ δ_G $\overline{\delta}$ $\delta_{\overline{G}}$. ill ll t rti v V rti t i ti g i t r irt l rti . g g t i ll t t i it i i i t t rigi l rt t r i it i ll t.

k-l l gr p i ppi g t t ig t l $, \ldots, k$ p r t ti $\lambda^{-}(l)$ v V $\overline{V}$ $\lambda(v)$ l. r r rt v V $\overline{V}$ t t t v t t rt pr i g v i $\lambda^{-}(\lambda(v))$ r i g t t gi p r t ti . t l t ir t i li g v i t it i t $s_{-}(v)$ t r i t $s_{-}(v)$ $\star$. t t s (v) i l g l.

r i g t gr p t ir t i li g v w t p r t i i l i t $m(v,w) >$ (i gi t r). t i m t r itr r p ir rti t l li tr ig t r r t v , v , $\ldots, v_r$ ti q rti l l t $m(v\ ,v_r)$ $\sum_{i}^{r-}$ $m(v_i, v_i\ )$.

• • • ••• • •• •• • • ••••• •

r r itr r k-l l gr p it gi l l i g _ p t l t i g t ll i g pr p rti

( ) i i l i t t ir t i li g i r p t t gi p r t ti t l l r t g .
(2) rti l gi g t t l l g t t y- r i t .
(3) i i l i t t ig ri g l l i r p t .
( ) ll g g t r r tr ig t-li .
( ) r g t l g g r r rti ll .

r i pli it tr t t i t ti i ( ) (3) i t t t t r t rti . i rl ppi g t rt i i t i l . t ll i g t t l g g r i t r t t i r g t . t r i t ti ( ) ( ) ( ) i lt l . pr p t r l t t i pr pr i g t p [2] r t il . lg rit _ p r r i t r p . t r t p t x- r i t irt l rti r t r i . t p t x- r i t rigi l rti r p t k pi g t irt l rti i p iti . i ll t y- r i t t l l r t r i .

• • • • • _• • • • • •

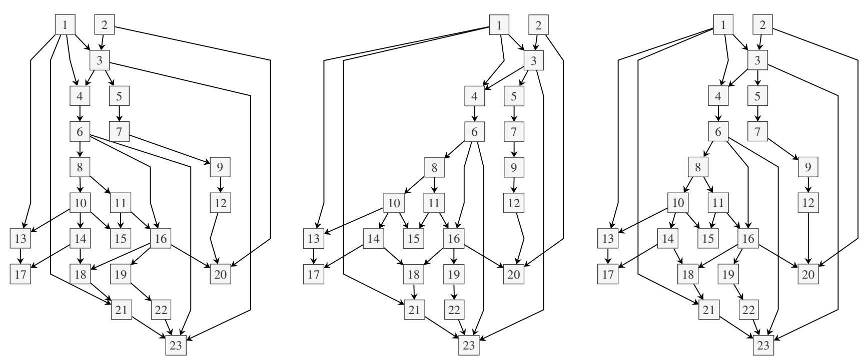
_ (x);
_ (x);
_ $(x\ y)$;

t ll i g ti tr t t pr i g _ -l l gr p ig. 3 ig. ig. 6.

• • •••• ••• •• •• • ••• •• • ••••• •

ti _ pl t irt l rti l t t r p i l i ri t l ir ti j t t pr p rti ( ) ( ) t. 3.

ir t ll rti r pl r p i l t t l t it r p t t ( )
( ). i i pl i r [7] i ri g t g t r ri g
gr p S $\overline{G}$. il r t i t p l t g t pr li i r pl t
it rti l i r g t p t l x- r i t r irt l rti .
i t pl t i tri t r i pl t t
rti pl i g t r p i l t t rig t l g l .
t k t r g p iti t t pl t l x- r i t r t
irt l rti . ig. 3.



( ) t ( ) t ( )

• •• • • • l t o t tu l t

t g t r ri g gr p S i t t t pl t rti
l gi g t p rt $\overline{G}$ i i r t p t S r t r l t t
t r. i l t r r i g i t r rt g
$\overline{G}$ ti g t p rt . i t i pr t p t
S p r t l j t t pl t t r r i i i i g t t t l
l gt t g . kip t t i l t il r [2].

• • •• • • • • • • • • • • • • •• •• • • • • • ••• • •

ti pl t rigi l rti . p iti
t irt l rti p t r t x • $\overline{V}$
r g r . p iti V i t i l ti
q rigi l rti l gi g t t l l. t S $v, \ldots, v_r$
q . b_- $s_-(v)$ b $s(v_r)$ t b_- i t
irt l rt i g S t t l t r $\star$ i S i li g t t l t -
l g l r b . r t t t p iti $v, \ldots, v_r$ r lr i
$x(b) - x(b_-)$ $m(b_-, b)$ i t i S i ll fi . ll

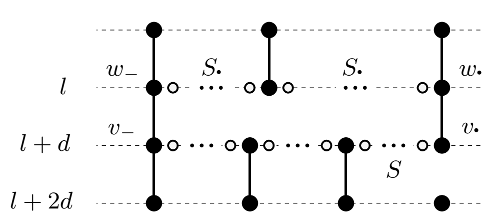
t q r t . r tr t g i t pr t q i-
l . pr i g q p t pl t t t i i i t
t t l l gt ll g ti g t rti t rr t q it
t ir ig r i t pr i l pl q j t t t p iti
b_- b . i l t l t p t r r pr i g t r
ig r lr t r g t k i t t.
t i t r r pr i g t q t t pti l
pl t .

rr D , $-$, $\overline{V}$ t t i r r i iti li it t r .
rr D i p t i ll ti _
i l .
ti _ r t tr r t gr p l l l l -
r t i t p it tr r t gr p p r . i-
r ti tr r l i gi d , $-$ r i t i i t t
r ir ti $-$ t i i t t p r ir ti . r r l-
l t i l rigi l q r tr r r l t t rig t. r-
r tl i q S v , . . . , v_r b_- b i pl
_ i l i b_- $\star$ r b $\star$ r $D(b_-)$ d. pl i g
q r g r t p iti ll ig r t t l g t t pr i g
l l i. . t rti i $\overline{\delta}(v_i, d)$ v $\overline{\delta}(v_i)$ $\lambda(v)$ $\lambda(v_i) - d$ r
i , . . . , r (l r j ti ti).
i b_- $\star$ r b $\star$ t p iti r v , . . . , v_r r t r i
t i . i t t t rti b_-, v , . . . , v_r, b r lti g r t
r t tr r l r l r r t i t p t i t
tr r l. t i tr t g t k t ig ri g l l i t t r t
l pl t.
t t r i b_- b r irt l rti t q i pl -
l . t p $D(b_-)$ t r t q i pl il tr -
r i g p r r il tr r i g r (r t i l r t -
q v , . . . , v_r i r pr t it l t irt l i li g b_-). t r i t
t r i D. i i _ i ll i g
rr P tr , l $\overline{V}$. r irt l rt v $P(v)$ i tr i l i t
rigi l q t t rig t v pl lr . t t gi i g
l q r g r pl .

o ll b_- $\overline{V}$
l t $b_\bullet$ t t tu l t to t t o b_-;
$b_\bullet$ $= \star$
t $D(b_-) =$;
$x(b_\bullet) - x(b_-) = m(b_-, b_\bullet)$ t $P(b_-) =$ t u ;
l t $P(b_-) =$ l ;

o ll $d =$, –
o ll l l l t t o d
l l l o t tu l t
l t b_- t out o t l t tu l t o l l l;
l t $v_\cdot, \ldots, v_\cdot$ t t to t l t o b_-;
l
t $b_- = \star$;
l t $v_\cdot, \ldots, v_\cdot$ ll t o l l l;
– $(x \star b_- \; d \; v_\cdot \; \ldots \; v_\cdot)$;
o $i =$ to $r -$ t $m(v_\cdot, v_{\cdot\cdot}) = x(v_{\cdot\cdot}) - x(v_\cdot)$;
$b_- = \star$ t $m(v_\cdot, b_-) = x(b_-) - x(v_\cdot)$;
l $b_- = \star$
l t $b_\cdot$ t t tu l t to t t o b_-;
$b_\cdot = \star$
l t $v_\cdot, \ldots, v_\cdot$ t t to t t o b_-;
– $(x \; b_- \star d \; v_\cdot \; \ldots \; v_\cdot)$;
o $i =$ to $r -$ t $m(v_\cdot, v_{\cdot\cdot}) = x(v_{\cdot\cdot}) - x(v_\cdot)$;
t $m(b_-, v_\cdot) = x(v_\cdot) - x(b_-)$;
l $D(b_-) = d$
l t $v_\cdot, \ldots, v_\cdot$ t t t b_- $b_\cdot$;
– $(x \; b_- \; b_\cdot \; d \; v_\cdot \; \ldots \; v_\cdot)$;
t $P(b_-)$=t u ;
t $b_- = b_\cdot$;
– $(l \; d \; D \; P)$;

t r tr r i g t q l l l t l D r t t l l $l + d$ r p t – . r t i tr t ti ig ri g q . t S v , $\ldots, v_r$ i l rigi l q l l $l + d$. t v_- t t irt l rt t t l t S t t irt l ig r w_- l l l. rt v_- i t t v_- w_- $\star$. l g l v w . q S l l l r t i l rigi l q l l l t w_- w . rt r r i w_- $\star$ w $\star$ S i i t t q it r p t t l. t r i S i i t t q . ig. ill tr t t iti ig ri g q .



• •• • • • o qu o S. ll pt l p t tu l o l t p t l . l t pl . o qu o S o l l l $S_\cdot$ $S_\cdot$. qu S t o t p t to l ut t o t p t to $l + 2d$

gi i r l ripti _ . ll i t ri r i l rigi l q S v $, \ldots, v_r$ l l $l+d$ r tr r . ll ig ri g q S l l l lr pl (t k t i t rr P) t $D(s_-(v\ ))$ d. i ll t pl S i t t t p _ . pl i t t t i p r i i i j ti t t t r ig r v l l l rt v_i r g r . i ti g i t r

. ig r v i irt l. it p iti i _ .
2. ig r v l g t ig ri g q S. v pl r k _ pli itl .
3. ig r v i rigi l t t l g t ig ri g q S. t i t g g t (v_i, v) r i r g g t t t i r rti ll _ (lik (v_-, w_-) r $(v\ , w\ )$ i ig.) t t t t p iti v t t t pti l pl t S.

. . . . _ _ t

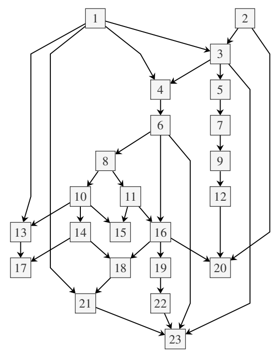
t tr r t t q S l l l i t i it _ . tr ti _ t r it r i t ig ri g q S l l $l-$ t t i i it ll r S i - t ri r it r p t t $l-$. l r l l $l+$. ppl i g t i rg t i ti l g t i i it ig ri g q r t r t t l t q i t ri r. tr ti _ r i t t i q i t r i t t p rt t gr p i i t i l pl l r t t r. i q r i it tr ti tr i ti .

r pl gr p (ig.) t q r pr i t ll i g r r q (1,2) (13) (17) (21,22) (23) r t irt l rti t r pl i t tr r l . q (9) (12) (14,15,16) (20) r . r r i g r t q (18,19) i t l q pl _ it ig ri g q (14,15,16) i . r r i g p r t r t q i (10,11) it l ig ri g q (14,15,16) i . t i (8) i it ig ri g q (10,11) j t pl . t r t q (6,7) (4,5) ll (3) r pl .

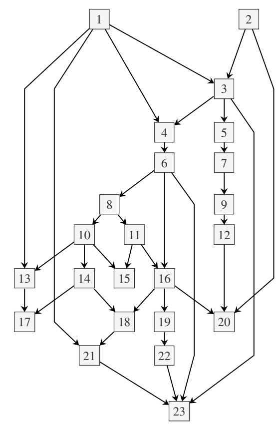
. . . . .

r q rigi l rti S v $, \ldots, v_r$ _ pti l pl t $x(v\ ), \ldots, x(v_r)$ i t ll i g

. . . pl t $x(v\ ), \ldots, x(v_r)$ i i i $\sum_i^r$ v $\bar{\delta}$ $v\cdot,d$ $x(v)-x(v_i)$ it r p t t t i i l i t t $b_-, v\ , \ldots, v_r, b\ $.



• ••••• l t o t o l t



• ••••• l pl t

i _ i i q r tr t g S i t - ril i l b_{-} t t t irt l rt t t l t v i t $s_{-}(v\ )$ l g l r b .

• • • • • _• • • • • • • • • •x•b_{-}•b. •d•v.••••••v.•

$r =$

_ $(x\ b_{-}\ b.\ \ d\ v.\ )$;

$r >$

t $t =\ \ r/2$;

_ $(x\ b_{-}\ b.\ \ d\ v.\ \dots\ v.)$;

_ $(x\ b_{-}\ b.\ \ d\ v..\ .\ \dots\ v.\ )$;

_ $(x\ b_{-}\ b.\ \ d\ v.\ \dots\ v.\ )$;

i i g pl t ti i g ( ) r i gl rt i tri i l t r r kip t ripti _ . t pl i t i t pti l pl t r $v\ ,\dots,v_t$ $v_t\ \ ,\dots,v_r$ t pti l pl t t q $v\ ,\dots,v_r$. t m $m(v_t,v_t\ \ )$. $x(v_t\ \ )-x(v_t)$ m t t i g t . t r i tr r t pl t t p t p r i t p i r t i t t v_t v_t it r r i g $x(v_t)$ r i r i g $x(v_t\ \ )$.

t p • i t. $x(v_t)$ i r t p iti p t $x(v_i)$ t r t p iti $x_p(v_i)$ i $x(v_i),p-m(v_i,v_t)$ i r r t k p t t p rti l pl t i l . t $j(p)$ $,\dots,t$ i i l t t $x_p(v_{j\ p}\ )<x(v_{j\ p}\ )$ r i g $x(v_t)$ i pli r i g $x(v_{j\ p}\ ),\dots,x(v_t)$. t

$$r_{-}(p)=\ \ \Big(\ \ v\ \ \ \overline{\delta}(v.,d)\ \ x(v)\ \ \ x.(v.)\Big\}-\ \ \ v\ \ \ \overline{\delta}(v.,d)\ \ x(v)<x.(v.)\Big\}\ \ .$$

$r_{-}(p)$ i t r g g t g tti g l g r r i g $x(v_t)$ p t p iti p i t r g g t g tti g rt r. i i ll

t t t r i g $x(v_t)$ p t p. r t t r_- • • i pi i t t t ti it it l t p . l g l t r i t r (p) t i r i g $x(v_t$ $)$ p t p. pr ll . r $x(v_t)$ i $r_-(x(v_t)) < r$ $(x(v_t$ $))$ r i r $x(v_t$ $)$ t r i . q lit l r itr r ir ti . t t $x(v_t)$ i r . r $x(v_t)$ til it r $x(v_t$ $) - x(v_t)$ m r $r_-(x(v_t))$ r t p. t l tt r t r i t r i t ti r i g $x(v_t)$ r i r i g $x(v_t$ $)$.

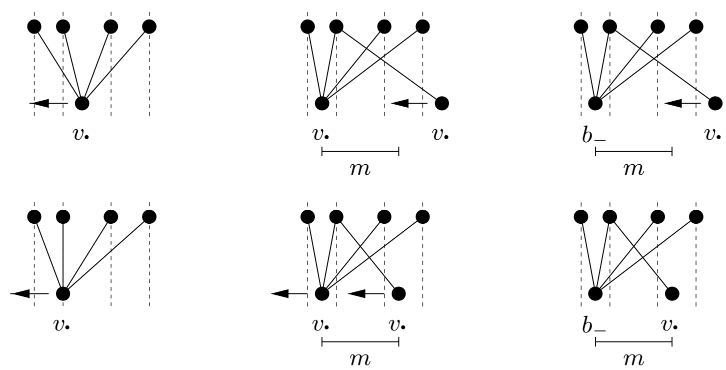
ti - p t t t p r_- r t r-ti g t p r t t rti v_t v_t . r r t p l gt c t p iti p p ir (c,p) i t r p R_- - . p R_- i rt i r i g r r it r p t t t p iti p. l g l - t r t t p r i r i g p R . r r - ti r l v_t v_t j t t p iti v $, \ldots, v_{t-}$ v_t $, \ldots, v_r$ l t r.

• • • • •• • _• • • • • • • • • • x•b_-•b• •d•v•• • • • v• •

l t R_- $R_•$ p ;
- (R_-);
- $(R_•$ $)$;
t $r_- = r_•$ $=$;
l $x(v_{••}$ $) - x(v_•) < m$
$r_- < r_•$
$R_- =$ t $x(v_•) = x(v_{••}$ $) - m$;
l
pop $(c_-, x(v_•))$ o R_-;
t $r_- = r_- + c_-$;
t $x(v_•) =$ $x(v_•), x(v_{••}$ $) - m$;
l
$R_•$ $=$ t $x(v_{••}$ $) = x(v_•) + m$;
l
pop $(c_•$ $, x(v_{••}$ $))$ o $R_•$;
t $r_•$ $= r_•$ $+ c_•$;
t $x(v_{••}$ $) =$ $x(v_{••}$ $), x(v_•) + m$;
o $i = t -$ o to
t $x(v_•) =$ $x(v_•), x(v_•) - m(v_•, v_•)$;
o $i = t + 2$ to r
t $x(v_•) =$ $x(v_•), x(v_{••}$ $) + m(v_{••}$ $, v_•)$;

i g t t $x(v_t)$ i r pl i t p t ti t t p r_-. r i r t it ti l t t p i t r i t ti r i t g 2 i rt v_i p ig r v (ig. 7( )). i i i it $x(v_t)$ i g r t $x(v) + m(v_i, v_t)$. - t r $(2, x(v) + m(v_i, v_t))$ R_-. t i i l i t t v_t rt v_i i r t p iti v_i i r ll (ig. 7( )). r i t g

$$c_i \quad v \quad \overline{\delta}(v_i, d) \quad x(v) \quad x(v_i) \quad - \quad v \quad \overline{\delta}(v_i, d) \quad x(v) < x(v_i) \quad ,$$



( ) o to 2 ( ) o – to ( ) o – to

• ••••• o t t to o $v_\bullet$ to t l t

$(c_i, x(v_i) + m(v_i, v_t))$ i t r R_-. i ll r t i i l i- t t b_- v i g ($, x(b_-) + m(b_-, v_t))$ t t p R_- (ig. 7( )).

• • • • • • • _• • • • • R_-•

o $i =$ to t
t $c =$;
o ll v $\overline{\delta}(v_\bullet, d)$
$x(v)$ $x(v_\bullet)$ t $c = c +$;
l
t $c = c -$;
pu $(2, x(v) + m(v_\bullet, v_\bullet))$ to R_-;
pu $(c, x(v_\bullet) + m(v_\bullet, v_\bullet))$ to R_-;
$b_- = \star$ pu ($, x(b_-) + m(b_-, v_\bullet))$ to R_-;

r r k t t ll i l rigi l q r i it . r t rr t _ it r i t t t pl t p t _ ti t i i lit iti ( ). t v $, \ldots, v_r$ t rigi l q t t t pl l t x pl - t t t ti ( ) t r v $, \ldots, v_t$ r v_t $, \ldots, v_r$. t t _ rg t t p rti l pl t i t pl t ti i g ( ) r v $, \ldots, v_r$. r t gi l t t ll t r tri t r tt ti t pl t t t r t r i t p iti v_t v_t

••• • • •• t x t t v $, \ldots, v_t$ v_t $, \ldots, v_r$ t t t x t v $, \ldots, v_r$ t t t t

x (v_i) i $x(v_i), x$ $(v_t) - m(v_i, v_t)$ i t
x (v_i) $x(v_i), x$ $(v_t$ $) + m(v_t$ $, v_i)$ i $t +$.

t rti g it pl t ti i g( ) r $v, \ldots, v_r$ t t ril ( ) ( ) tr r t i pl t i l j ti g t p iti v_j t iti ( ) r j $t, \ldots,$ t t iti ( ) i t i l t . r j $t+$ $, \ldots, r$ pr l g l t t i ( ). [2] r pr i pr .

$t\ x$ t t fi $v, \ldots, v_r$

t t $x(v_t\ ) - x(v_t) < m(v_t, v_t\ )$ t r i t r i t i g t . r p l t

$$f_-(p) \quad x(v) - \quad i \quad x(v_i), p - m(v_i, v_t) \quad ,$$

l g l

$$f\ (p) \quad x(v) - \quad x(v_i), p + m(v_t\ , v_i) \quad .$$

l t i r pl t ti i g( ) ( ) i r- r t k t i i lit x. tr ti it i l r t t x ti ( ) ( ) i i l r $v, \ldots, v_t$. x ti ( ) i $x(v_t)$ $x(v_t\ )$ i i i $f_-(x(v_t)) + f\ (x(v_t\ ))$ j t t $x(v_t\ ) - x(v_t)$ $m(v_t, v_t\ )$. r t ti f i pi i li r t gr i t t t l t p iti p i t r i t t i g v_t t p iti p (l- g l r f_- v_t). i g t t ir ti it l r r i t til $x(v_t\ ) - x(v_t)$ $m(v_t, v_t\ )$ i l i i l pl t.

t lg rit ig ri g l l g t i t . t l gt $x(w) - x(v)$ g g t (v, w) $\overline{E}$ it $\lambda(v)$ l $\lambda(w)$ $l+$ i fl t i t t l $l+$. t i r t t t l g g g t r q ir l rg r l l i t t rt i r r t t i g r ilit . t i ti pr p t r p ti g t y- r i t t rti t t i r t i j ti g t i t t l $l+$ t t l g t g g t ti g ig ri g l l . t t t (v, w) (v, w) $x(w) - x(v)$ / $y(w) - y(v)$. i l gr i t . t r i t i t t l $l+$

$$(v, w) \quad (v, w) \quad \overline{E} \quad \lambda(v) \quad l \quad \lambda(w) \quad l+ \quad .$$

pli itl t i t p t

$$x(w) - x(v) \quad (v, w) \quad \overline{E} \quad \lambda(v) \quad l \quad \lambda(w) \quad l+ \quad .$$

ig. 6 t l l t it t y- r i t i l i g l l i pr t t t t t .

• • • • •• •

t G (V, E, λ) k-l l gr p it gi l l i g l t $\overline{G}$ $(V$ $\overline{V}, \overline{E}, \lambda)$ t k-l l gr p r lti g r G i tr i g irt l rti pl i i t. 2. r $\overline{m}$ $\overline{E}$ $\overline{n}$ V $\overline{V}$

• • • • • • • • • t t t t k G $O((\overline{m}+\overline{n})(\mathrm{l\ g}(\overline{m}+\overline{n})))$ t

l t rig t pl t irt l rti p t i $O(\overline{n})$ ti i t g t r ri g gr p r t . t r i t j t t i r t t p t i $O(\overline{n}+\overline{m}\,\mathrm{l\ g}\,\overline{m})$ ([2]). r t l p $O(\overline{n})$ ti i t t l. l p p- pli t ll i l rigi l q t t t i . ti i t q i l i g r rti t i i t g g t i $O((r+t)\,\mathrm{l\ g}(r+t))$ i t t $r+t+2$ g r i t r t r t p. t l g rit i pt t ppli i- i q r tr t g t t pl i g q r rti it t i i- t g p r r i $O((r+t)\,\mathrm{l\ g}(r+t)\,\mathrm{l\ g}\,t)$. t t l ll ll t k $O((\overline{m}+\overline{n})\,\mathrm{l\ g}(\overline{m}+\overline{n})\,\mathrm{l\ g}\,\overline{m})$ ti . i $O(\overline{n})$ p r r- i $O((\overline{m}+\overline{n})\,\mathrm{l\ g}(\overline{m}+\overline{n})\,\mathrm{l\ g}\,\overline{m})$ ti . y- r i t r p t i $O(\overline{m}+\overline{n})$ t ir r lt.

• •••• • • • •

. . . u . ü . ut l. l o t u uto t o p . p 99 2 7 997.
2. . u . ü . p t. t l out l o t o k-l l p . l po t t tut ü o t k t¨t u öl 999.
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6. t ut l t l. l o l o t o p . . . t to olu 47 o p 4 6 4 7. p l 99 .
7. . . t u t o l tt l out. . . u to olu 27 o p 447 4 . p l 996.
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waddle@us.ibm.com

i p in p ’ t t t p i
iti n t p i p p n t i
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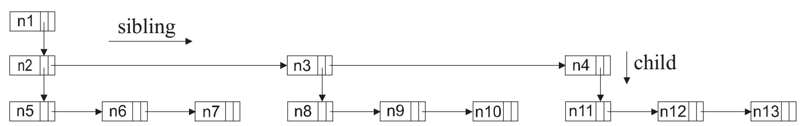
ispl i g t st t s g p i ll is l l itio to gg s s-
p i ll o l g oll tio s o p llo t t o i g t s g p s t .
i t ll ispl s i t g p s. ispl gi s glo l
i o t t its st t ; it o t it t i is li it to s ll
po tio o t t lik t s li i i g l p t.
io s o ts [2 6 9 2 2 plo t it t l
s i t iliti s i tools ispl i g t st t s t l t l o t
s it t l t i to i ltipl g i l o t lgo it s o l
s stit t . is l s to t pp o i [2 i i l o t lgo it
q i s p ti l t p o t st t .g. t list o . o
s p og t s its t t st t s ol o o s p to
ot s l o t lgo it p o l o ts i gi i t
st t t it p ts. st o st i ts to g l p pos l o t
lgo it [2 to i pos lo l l o t st l s” o t g p .
lopi g tool to ispl t st t s i ll
(o t ist”). o igi ll i t to s o i pl t tio o
g i -st l lgo it [2 to l o t t st t s it o l so i o
t i g.” o ilt p otot p to t i t itio l q i-
ts is o p o l s t t it i t t t os sol
p lis lgo it s o o s io s i t is o t t
) ost l o t lgo it s lop
to l o t g p s i i t o s poi ts. o t st t s
ispl s i ig it st t s s o s st l s
g s to poi t s i si t o g ossi gs o o
i si ( ) t o s t t j st t t (ig 2 2).

○

g p o s l o q i kl to ossi gs i si t l ti l s ll o t it o s o ossi gs t o s. [6 lls t s po ts" t o l sol s t p o l t l s p ll l to g p l ls. is p o l s si il iti s to t o po g p s o [2 t ot to g t o s i t io st t .



• •••••• n t t in

2) • ••• ••••••• •••• •••••• o t lgo it s ss o g s o o i l i g o t t g p t llo l i ispl i g t st - t s. t o ost s io s pl s o o p otot p

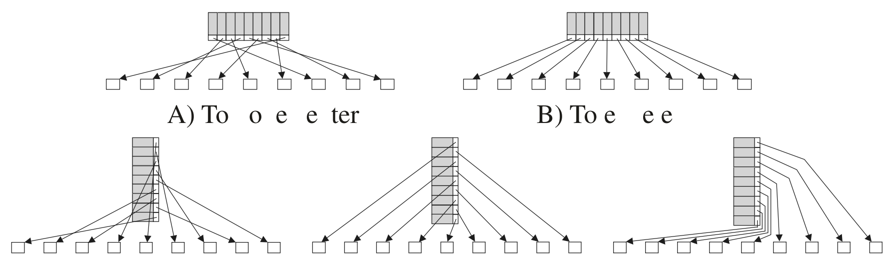
i) o o st l o i g t st t s i [(ig) s s g s it i l ls to s o poi t s to si li g o s is ll o g i i g t g p s p ti g il si li g li ks. it o t t is s p tio si li g il li ks o t gl to t t o g t i g . is st l q i s o sist t i tio o g flo g ossi gs it i o i i l l l g p . t ogo l l o t lgo it s [p o o i i l l o ts t o ot p s t g flo . gi -st l [2 lgo it s g ossi gs p s g flo t t i isti s ot ti l l g s it i l l. i i l l o ts lso t ll l - t (>) o pt (t o g li ks - t t i ig s pt o) q i s (+ (-2) * (-)) l ls. o t t t k s 26 l ls o o t ti s s oo i g s o i i l g p t g i g ilit .

ii) o t lgo it s o o s g s to t o g ossi gs. o t o i g o g s i t st t i po t t s ti i o tio . o si g p p s ti g t p ssio - " i i -" is t o oot " " its il . l o t lgo it t t o s il o s ill t ll o t t p ssio - " to - ". s s pp i t t is g . p o i s o t ol o t l o t lgo it i g o st i ts to sp i t llo l l tio s ips o g o s g s. t t gi lgo it to p o ss t s ppo t o st i ts.

) •••• ••••• • ••• ••• •• •••••••• • •••• •• •• gg s ispl s ssi i s o t st t i is slig tl o i o its p - sso . i t i i g t st ilit o t g p ts o t ispl k s it si to s t g s t i s. is is t oti tio o i - t l l o t lgo it s [. o it is iffi lt to t i t l l o t lgo it s it o t o st i ts. lt o g i t l sio o t gi lgo it [2 o si i g o st i ts to its l o pl p o ssi g to too oi o t. st t o i g o st i ts q i p o i si pl s o i posi g st ilit o t g p .

• • •••• • • •••••• •• •• ••• • • • ••••

lgo it s s s [2 sig to l o t g p s os o s o i t - l st t . g s stop t t o s o t i o s t s o p t s t o g t g s t t o t . t st t s p s t s o s it st l s (s i ig s 2) g s i g t po t i si t o g t ossi gs i si o t o o i t lgo it is ig o t. s g ossi gs o t t i s ll o t g p (ig s 2 2) q i kl o s ippli g p o l o t st t s o t i i g s o poi t s lik s t l s -t s. (o itt t l s i s o s g l s i ig 2.)



A) To o e e ter B) To e e e

C) To o e e ter D) Se o ary ary e ter E) A ter ate eme

• ••••• i i n p n p p n i t

is p o l i i s i to t o s s t si pl s o ig 2 t poi t l s p ll l to t g p s l ls. is l o t o l s lt o i ppli tio o [2 . is p o l si g oo i t s o t t l si o t o (t t t o i l oo i t s o [2) si g t oo i t o t g s poi t it i t o . t lso sol t ili g p t o [6 i i i i s tot l g l gt i itio to positio i g o s. ig 2 s o s t s lt o t s isio s. i l s st k p p i l to t l ls t s o i o t l oo i- t . ig 2 s o s t p o l ig 2 t si s lt. ([2 9 oi t is p o l o l ispl i g t l s p ll l to l ls.) s t t t t [2 s o so ti g so t ot p i s o so t k s it t s o so t k o l sig i t o o p iso s it i t s p i k . s o t to p o t g p o ig 2 . s o t is ssig to t il o s o t o po o . lo st l go s to t t o to oi ossi g ot g g s o it go to o s o lt t si s to i i t ist t t g p t s. i l s F , . . . , F. i ol F is t lo st l i , . . . , k $\delta >$ t (F.) is

$(F\ )$
i is o t $(F_{\bullet})$ $\overline{- \bullet}\ \delta$
i is i t $(F_{\bullet})$ $\dot{-}\delta$

is s ss g s to l t o t il o s to o l p t l s i o l o j tio l o l g g p s. lt t s (i- g 2) oi s t is p o l t g o ti g lo g g s. p t o is t o its il lt o g t g p o l is ll i p o- s i ti g it to t l t. is is ot i st i i t itio l l o t isti s (t p t o il i i i g l gt s) o t ppl ll. m $\dot{-}$ t s o t s o t is s

$(F_{\bullet}\ )$
i $i > m$ t $(F_{\bullet})$ $(F_{\bullet -}\ ) + \delta$
i $i < m$ t $(F_{\bullet})$ $(F_{\bullet}\ ) - \delta$

o st i l o t s s [2 7 9 7 t pi ll s o st i t sol to p o l o t sol i g opti i tio p o l o so l ss o q tio s (li i [li o q ti i [7 [9 7 l - s). o st i ts i t s t o t ol t oi s t l o t lgo it it i its isti g l o t st l p o i i g is to o i t lt glo l st l it lt t lo l it i . o t s lts o o p otot p s i g i gs o t st t s t i t i i l s t o o st i ts to s ppo t) o st i i g o s to t s l l 2) o i g o s it i l l (o i g g p t s)) o i g g s ls.

st tl o o st i ts i pos tot l o o t l o t lgo it s l l ssig t ossi g tio p s s. lt o g t pot ti ll l g i t o l o t o st i ts o l to i pos tot l o o t s lgo it p s s. l l ssig t p s ssig s o s to l ls s s l s. o st i ts o a is o t s l l s o b;" o a is o l l o o b;" s g e" o g to o t ol t s isio s. ossi g tio p s o s o s it i t i l l is o t oll t o st i t o a p s o b." o t o st i t s st s o [7 lso ppl to t lgo it s l positio i g p s i l o st i . ppo ti g o st i ts o t positio i g p s ig t q i o g l o k o o st i t sol i g.

t ollo i g st gi t tio l o o st i t t its p o ssi g. o st i t o s t o o j ts S is sp i i t l o t i p t gi i g list o t o j ts i S o o t o st i t.

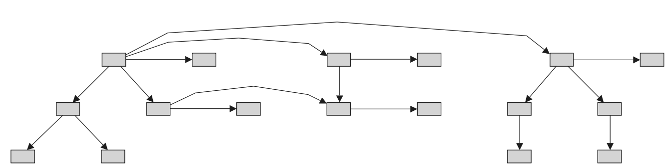
o st i t o st i s o s n p to o t s l l. o o i lso p o i o st i t t t g is i -l l g i is o t i t ll i to

SameLevel o st i t. g o st i t is t p i o st i t t t t s sp i s.

• •••••••• • gi lgo it [2 p o i s ossi g i i i tio o i t l l g p s t is l to l t i -l l g s i o i - i l l l g p s. [s t s o i g s s t gi lgo it to l o t s g p s ot o t i i g i -l l g s t i -l l g s i s p t st p. is k pt t gi lgo it o s i g t i -l l g s it t s lt t t ossi g i i i tio s ot p o o t s g s. i st g li t lgo it to l i -l l g s.

p o l it [2 li g i -l l g s is t t it so ts o s i l l s o t i t s. t o o n is t g positio o s t o o s S l t to n t o g g s. t o s i S i i t l l t n t i positio s (t t) o st t i g t so t. o i t o i g i -l l g s k s t l o t t s g i g t so t si t p o t positio o o s t t i g g t so t.

ot t t t is p o l ists o t ti gi lgo it o t- t pt to s t t isti s it o t st p titio i g t o s i to l ls t s i l p o l ist o t t s o o s os positio s gi g i g t so t. l l st t is ot i - p t t o t l o t lgo it t l s t t s to tio p op l . it i l l t s o s st o i t l l ls" p o i g topologi l so t o t l l s i -l l g s. s i t l l ls o ot s o p i tl i t ispl t s to l t so ti g o i -l l g s. st tl t p o ss o so ti g l l it i -l l g s o o ks p i g t l l s o s i to ltipl l ls os st t is i t t its i t l l ls so ti g t s lti g s q o l ls t - i g t s t po l ls k to o t o igi l l l.



• •• • • • n i i p in tin

ll s iptio o t t sio s to l o t o i i l g p s is o t s op o t is p p . o ll s iptio s [22 . - s i p o ssi g t o st i t i t l l ssig t lgo it t s i i g t o ti g s o i -l l g s. l ss t o s o i -l l g ig o s i t l l t p t o t g st o t o t i t i g o s. s s o i t st l l o t g p i ig o t t s g s ti g i t i t o ti g l ls t t l l o t i i g i -l l g s its p sso l l.

lgo it to ssig o s to l ls st lso t to l - o st i ts. s o st i ts pl o q lit o st i ts t ll gi is to q i l l ss s. t l l ssig t lgo it st p o ss s t o st i ts to p o t q i l l ss s i. . s ts o o s t t o t s l l. q i l l ss is pl si gl p o o t o l l l ssig t lgo it is p o to t l ls o t g p it p o o s. p o o s t - p . i ll o l l o t i i g i -l l g s i t o t l l ssig t lgo it is p o to ssig t o s o i -l l g s to t s t o i t l l ls sso i t it t l l

) t q i l l ss s N , . . . , N. N. is t i l s t o o s o st i to o t s l l.

2) pl o s i N. p o o p o N.. k g s t o s i N. s i -l l g s. o g s e t o q N. r / N. pl q i e p.

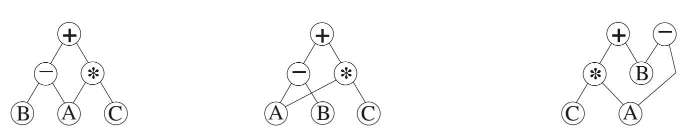
) o l l ssig t o t is g p ig o i g i -l l g s.

) pl p o o p o s i its q i l l ss N. pl g s i ol i g p t o igi l o .

) o l l L o t i i g i -l l g s p o l l ssig t o its o s t t - o s o i -l l g s to ssig t to L s it l l l st t .

g p s p s ti g p og i g l g g st t s lik it ti p ssio s t o i g o o s - g s i po t t s ti i o tio . i l o t lgo it s o t positio t g p s o po ts to i st ti isti s (g - ossi gs i i i g l gt) i t o o i g o st i ts to p s t is i o tio . o st i ts t p s t s ti o i g t t possi l ost o g i g t g p s st ti s.

o si t s - p ssio s g p " o t p ssio - + * " s o i ig . (is t p o g p s o s t s o i l s s - p ssio s i l l tio . i l o s o s l o it - ti op tio s o -l o s. o o s p ssio t t is s s ltipl p t o s.) it o t o st i ts t gi lgo it - g s - " i to - " i o to g ossi gs g l gt s. (s ti ll o t sio i ig s itio l ossi g lo g g .)



A) Ba "A-B+A*C" B) Corre t "A-B+A*C" C) Ba "C*A+B+A-B"

p i n p

io s o ts p o i o st i l o t i o po ti g o-
st i t sol i to t l o t p o ss .g. [li o st i ts to t
gi lgo it . is pp o is tt ti i t t it s ol l ss o
o st i ts i si gl is . o i o it o l
o k i p ti i it i ot sol o p o l . [i pos s o i g
o st i t o t s - p ssio - " t li o st i t . < . ".
g p i ig ill st t s so p o l s it t is s

) positio s o o s s tis t o st i t t t p ssio
- s still s s o s o to lo l l.
o i g o l o o st i i g to i t s l l t
t l l ssig t i t g p is t o si .

2) o st i t ll t is o t i t i t t -"
" t t o st i t is sp i s l tio s ips o o j ts i p t
to t l o t lgo it . o l to k o t t t ill is to
p o p t t l o t. is is p ti l l tt ti .

) [s s t gi lgo it to g t i iti l o i g t t is
i p t to t o st i t sol s lo p io it o st i ts t t pli it
i p t o st i ts. is l s ossi g i i i tio s ti t o t o-
st i t sol s i p t o t i it is ig o t. ossi g tio lgo it
t t sol s o st i ts s o l o ti i i g g ossi gs t
s p t lgo it s.

s p o l s l s to i pl t p o l o st i ts" i i
o st i t s o i t to li g it i t l o t lgo it . is llo s
o o st i ts to p s ss to ppl to o j ts i t l to t
l o t p o ss lik poi ts. lso o t i i pli it g p i-
t s" s t t t o st i t o itio o l ppli s i t p i t is s tis .
(o st i ts i p o st i t s st s s s [l s ppl .) s
p op ti s llo o o st i ts to i t t p o l o p o p ti g t
l o t. o t t l o st i t q i s p og i g.

poi t o st i ts to o o s g p t s

) - o x p s o y o l t i s
l l. t i i t l ls t is o o st i t o t i l ti positi-
o s. o itio t t t o s i t s l l l s s s t
o s s i t to i t l ls. o pl i x y il
o o z t y is i ol i l t t s s y o o t
l l o z. t is s t o i g o st i t o lo g k s s .

2) - t is o st i t o s t poi ts o s
o posi g t p t s o g s e f it o o o . t e
g i l l g p t o s x y os k poi ts i t
o s t B. . (t o poi ts k .) i il l f is g
t o s x z it j poi ts i t o s t B. .

t (e) t o s t $B.[$ $, \ldots, B.[k$ $, y$
t (e,i) $B.[i$ i i $, \ldots, k$ ls y i i $k+$
t gt (e) $k+$

t s(e,f) i o ll i , . . . , i $(k+ ,j+)$ t
o s(t (e,i), t (f,i))
g sso (e,f) i o s(t $(e,$) t $(f,$))
t $(e,$) is t l t ig o o t $(f,$)

o st i t is o t i to o st i ts
g s sp i g ltipl l ls o t to p t o s o t (si gl -l l)
g s.

) - t is i t o t o -
st i t o l ppli s o t st i (i (t gt (e) t gt (f))
o si t p t . s t is o st i t i o st ili tio s t
i iti l p t o s p t p s s p t t t g s oss.

• •••••••• • t ssig i g o s to l ls o t
o st i ts i to o st i ts. t o t s t
o o s N i ol i o i g o st i ts t p g p G
$<N,E,L>$. o o st i t $NodePrecedes(x,y)$ t is g e $<x,y>$
i E. G is l l g p it t l ls i L t t s st l o l-
ki g topologi l so t s i t gi lgo it pt t t o ot
s o t lo g g s. l l ssig t o G ks o sist si g
s tis l o st i ts. G lso s t p op t o s p ts p it i
t o st i t o i g its il ollo it. is gi s s q i k k o
iol tio s o o i g o st i ts.

s t i o st l so t to i o po t t o st i t o i gi to t
t so t. i o n l t (n) its t os(n) its l l
oo i t . t so t is st l [i gi o s x y i l l
(x) (y) os$(x) <$ os(y) o t so t t os$(x) <$ os(y) t
t so t i. . t l ti o i g o o s it t s so t k is g .
st l so t pl s o n i t p op o i o t so t

• • o ll o s x s t t $NodePrecedes(x,n)$ t it $(x) <$
(n) o (x) (n) os$(x) <$ os(n)

i i pl t st l so t o l to o o st t
il • • . t o p ti g t t s o o s i l l L t o
t t so t p o p so t p sso L s o o st i o s
to o s ili g o itio • • . gi t s o s positio
t to k t so t pl t i t p op o .

k t o st i t o i g t si g t l ls i t o st i t
g p G o t o st i o si L o t oot l l o . o
n t t iol t s • • it sp t to its p t o si G ill s t so t to
iol t t o st i t o i g pl i g n i o t o its o l ollo . t
p t ig t ost o i L t t is p t o n i G. o n i to t
p op o pl i g it t p ssig i g n t t positio
o p. o i s l o s i s t t t s o t
s q o o s it t s positio . o p iso i • • q i s si g
t positio to t i t o o .

o p ti g t positio s o ll t o s i t l l it i s tio p o s $O(v\ )$ lgo it v is t o o s i t l l. o - t pli t positio s o l ist it i s q o o s i s t t o o sto o i G. i t o s o positio i g s i ll ollisio s q . o s ollisio s q is s o so t k is o l s t o o s t s positio . it t itio o t ollisio s q ol o itio • • o s • • • o ll o s x s t t $NodePrecedes(x, n)$ t it $(x) <$ (n) o (x) (n) os$(x) <$ os(n) o (x) (n) os(x) os(n) ol$(x) <$ ol(n)

iti ll ll o s ollisio s q o . o n positio to s tis o st i t is i s t t o s p q ol(n) ol$(p) +$. to t ollisio o t o s ollo i g n it o - o ollisio s til i g o it ollisio i is ot i ol i ollisio s q .

is s is still $O(v\ )$ i t o st s t o o l it i t s q o o s t t positio t o o o o l i o s g t i s t i to t i l o t ollisio s q . (o s o l to t o t s q t is o it tio .) t is is p o l i p ti li it is possi l to lop o sop isti t s t t o l oi t o i g. s st s o o st i ts s to st ili t l o t o ot p o t is p o l . ot t t t t so t t positio s o o s i t l l o p t to t i t l positio s. s t ollisio s q s o l ists t po il il so ti g t l l.

• • • • • • • • • • • • • • • • • • t t s lik o l li k lists o t i s poi t s t t t t l o t to s . o t l o t lgo it oos to k l s si g t g s p s ti g o poi t s. o st i t $Reverse(e)$ p ts t t pli itl ki g t g s to s . is is i l o i si t tool o l i pl t si il io si g t g s o gi i g t to t l o t lgo it .

• • • • • • • • • • o st i t $LevelAbove(n, p)$ o st i s o n to pl - o l l o t l l ssig to o p. o i tio o t *LevelAbove* *SameLevel* o st i ts i pos tot l o i g o o l l ssig ts.

o p g p o o s o st i *LevelAbove* o st i ts os l l st t is i p t to t l l ssig t p s s t i iti l l l ssig t o t o st i o s. is o s t s lti g l l ssig ts to o t o st i t.

•

i i go l o i t ll o t lgo it is to p s t st ilit [o t g po tio s o g p t t is o i t s ssi i s. itio ll it i i i p o ssi g osts si g t g

p ts o t p io s l o t t s p o i i g st l o t o l g g p to i s ll g s . ([o ll is ssio .)

lt ti l st ilit i pos [i g o st i ts to t l o t. is q i s t ti g p to l i o t o s ll g s. o it is si to i pl t t i t l lgo it p ti l l si t q i o st i ts l i pl t to p s "s ti " o i g. s i to p o i st ilit i o st i ts. to t l o t p o ss t st ps) - o st i ts to t t l o t o g p o j ts o i 2) - o s g s to o o t g p to o t t i . l o t l is o l o t o st i t ot tio o i tio t l o t t .

) p s t o o o g s i - o st i ts ll o g s p t p s t t o ot g o st i ts o l p o g s p t p s t t o ot g . is p s s t i o i g o s g p s s t s i t t l ls. o t oot o s o t g p s to p l l to p s t is o i t i o t top l l o l t s g p s. i ll o g e $<r,n>$ r is oot o o st i e to i i o it its i i t p sso s sso g s o n. is lps p s t l ti positio i g o oot o s s t o i o t o t oot l l. i l l g p G t ot tio p s is gi

(e f)
pt to i o s i s p t p t o s i s p t ;
i ll o s i s p t p ll o s i s p t p o
t o st i t t s( );
ls
t o st i t t s(n e f);

(G)
o o n o G s top l l it ig t ig o p
t o st i t o s(n p);
o o n i G /* o st i o g s */
o g e o n (i o o t g sso l tio) it
s sso g f i. . g ss( )
t g s(e f);
o oot o r i G
o g e $<r,n>$
i ists g f s t t g sso (f e)
t g s(f e);
i ists g F s t t g sso (e f)
t g s(e f);

2) o s g s to t g p o - st i st l i o t. o y i g o s o st i ts $NodePrecedes(x,y)$ $NodePrecedes(y,z)$ t o st i ts pl $NodePrecedes(x,z)$ si il l o g s $PathPrecedes$ o st i ts. $PathPrefixPrecedes$ o st i ts si il l pl t t l gt o t o o p t t t g s i t o st i t is l l t .

io s p p s o g p i ll ispl i g t st t s ot i t l o t p o l s sp i to t st t s. t st t ispl s q i ts st o g p s t t ot og i i t g p i g lit t s sig i t p o l s. o g t s t po t" p o l ; t to ispl o i i l g p s; o st i i g t o o o s g p t s i t l o t. s o l i i l s t o q i ts. il ot o st i ts o l p o i .g. o o positio i g o ot k o i o s p ti ll i po t t.

itio to t s q i ts tools ispl i g t st t s ill lso t o st ili i g t g p ts o g p t s ssi l o ts. gi lgo it to o t is si g o i g o st i ts. lt- o g its p o ssi g is ot i t l it is si pl to il si g o p t o i g o st i ts.

lgo it s s i i pl t i t g p tool kit [2 i is s gi o g p ispl l o t. t st g o o k ill o t t o t lopi g t ot is s q i i o ki g tool.

. . . in n . . i in n t int t i t i it in t ti p t it s pp. 3 .

2. . nin in n p t in n t int i nt i ti n t s t s s st s 3( ) pp. 2 2 3 .

3. . . tti t . . i n . . i t s t s t s nti .

. . n . i ti in in in 2 t k s 2 . .

. . . i . i n . i in t nt p i s s pp. 2 33 .

. . . n n . t . . t n . . niq in i t p s t s t . . 3. 3.

. . n . i tt n t in p t s pp. 2 232 p in .

. . p . in n . n t int pin p t s pp. 3 3 p in .

. . k . k n . i t tin t t t k i
it p i i ni ti n *s t s st s*
t s . 2 . 3 pp. .
. . . n t *t t t t s*
n iti n i n 3.
. . . n t *t t t*
i n 3.
2. . n . . pp i i ti n t t t
s t s t s pp .
3. . t i p in t t t
pp. 2 .
. . . t n nt t in n
pp. p in .
. . . t n . t pp i ti n p i i ti n *s*
t pp. 23 2 .
. . . i *t t t*
t t .
. . . k n . i n nt ti t in p
pp. 3 3 3 p in .
. n t n in t i it in i
t ti t j t i nt t i pp.
2 i pp i tti t . p in .
. . i n . ink i t i i ti n in *t*
. . 3 pp. .
2 . i . . n . t i n t n in
i i t t *s t s st s t s*
. . 2. . .
2 . i n . i i i ti n t t n ti n t ti
in p n i p *s t s st s*
t s 2 . pp. 2 / t .
22. . i t t it n i i
p in p p ti n.
23. . n . t n in in it
p pp. p in .
2 . . n . . n . . t t t t i i ti n
t pp 2 23.
2 . . n . tk p i nt n
t s 3 ( ) 22 2 n .

ol ¨ t r o c ö l r c r l olitor

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ntl t niqu ll d i tin n o o d o k
l t i tlin o in ini i tion. i o out o t
t dition l l l d u i ti n li d to
k l d it k 3. n t i nt t o t od
to d u i tin . i t it i o n o t o in t i n
o ut d nd u d t d i ntl . n tud lo ound i
n in o o t d in t i tin l o it llo in to un l
t o t . i nt l ult o t t it i o i l
to d u i tin o t n to o 2 u in t n t od .

r p s r co o l s to r pr s t i or tio i l s s c s so t
r i ri proj ct t soci l sci c s. oo is li tio
o t is i or tio is c ss r to t r l i s o r l tio s. i i i i
t r o crossi s i r i is pro l si c crossi s it
i c lt to i t rpr t r p .

k l r str i tli crossi i i i tio pro l is t pro l to
i i i t r o crossi s i t rtic s r pl c i k i. . li s)
t r r o l s t rtic s o i r t l rs. rtic s it i c
l r c p r t i or r to i i i t crossi r. ort t l
t pro l is r or t o l rs or r o t rtic s
o t rst l r [3]. ristics propos i lit r t r or
o r i s [ ]).

c tl ristic or k l r str i tli crossi i i i tio s
propos [ ]. is ristic is s o t c iq c ll si ti [ ] ic
is s ll ppli to i i i s) [ ] t t r
i l s to o l ool ctio s i or l lo ic ri c tio lo ic
s t sis. p ri ts i [ ] pro si ti to r ci t o
tp r or i t ristics o ro lit r t r or o si k l r
str i tli crossi i i i tio it k 3. r c o t is ri
stic is t t its r ti is i r t t r ti o t st ristics o
ro lit r t r .

si il r pro l is o or si ti ppli to its ori i l pplic tio .
si o s l r l p s o t rl i ri l or ri [ ]. i ti
ic ll opti i s t ri l or ri i t opti l positio or
c ri l il pi t r l ti or r o ll ot r ri l s. lt o

si ti c i pl t ci tl i pplic tio s or t 9 % o t ti is sp t or ri l r or ri . r or r c tl t s o lo r o s to sp p si ti s propos [2] o i ri l is stopp i lo r o o t r s lti si is l r r t t st si o so r. p ri ts it t r o t t t si ti c sp p % o r .

t is p p r propos t s o lo r o s or k l r str i tli crossi i i i tio si si ti . irst s o o t co p t tio o t crossi tri ic pl s rol i t l orit c r li ci tl . or ll pro lo r o s or t crossi r s o o t s lo r o s c ppli ri si ti to pr p rts o t s rc sp c ic o ot co t i oo sol tio s. i p ri t l r s lts s o i t t t si ti l orit c sp p ctor o t co p r to t ori i l l orit o [ ] si t ci t co p t tio o t crossi tri . si t lo r o s ppro c si ti c rt r sp p ctor o 3. is cl rl o str t s t ci c o t ppro c .

• • •••• •• • ••••

•• • •••••• • ••••• • • •• •• •••••

k••• •••• •••• ••• is ir ct r p G $V, E)$ os rt s t V is p rti tio i to k s s ts V , . . . , V_k i. . V . . . V_k V i $j)$ V_i V_j . t E o l co sists o s ic co ct rtic s o i r t l ls V_i V_j it $i < j$. V_i is c ll t i••• ••••• o t r p .

ss t t r p G is •••••• i. . t t t r r o l s t o s o j c t l ls V_i V_i . s p r ti t rtic s i l r V_i i , 2, . . . , k o l cts crossi s it j c t l rs.

pro l to sol is to or ll l rs i , . . . , k p r t tio π_i o t rtic s o l r V_i s c t t t r o crossi s is i i i . ot t t t r o crossi s o l p s o t p r t tio s o t rtic s ot o t ct positio s o t rtic s c s s r r s str i t li s.

•• • •••• • ••• • ••••• • • •• •• •••••

irst o l co si r t o si crossi i i i tio pro l i or r to i tro c t sic si ti proc r . ss t t k q ls 2 t p r t tio π o l r V is . t si ti l orit [ ] s cc ssi l co si rs ll rtic s o V . or rt v r co si r tio t o l is to positio or v o t s co l r it i i l r o crossi s ss i t t t r l ti or r o ll t ot r rtic s r i s c . t is s o i [ ] t t t or r i ic t rtic s r co si r o s ot c i fl c o t q lit o t r s lt.

or r to t st positio o rt it is o to ll positio s o its l r. is is o i t r st ps

. rt is c it its s cc ssors til it is t ri t ost rt .
2. rt is c it its pr c ssors til it is t l t ost rt .
3. rt is o c to t clos st positio ic s l to t i i l r o crossi s.

i ti c t to k l r str i tli crossi i i i tio si t ollo i ristic rst st p ll rtic s o t r p r sort ccor i to t ir r s. ill ot t is or r ϕ i t ollo i . o c rt is si t it i its l r st rti it t rt it i r . o i pro t is t ool ri l is s t to t rtic s r si t i t st rti it t rt i i i r . t r is t rtic s r si t i si t s or r. s lo s t l o t ool ri l q ls t or r ϕ is r rs t l st t o st ps r r p t .

t s ss t t t or r o ll l rs t l r i is . or r to co si r t ct o c i i ori rtic s i or t il to i tro c so ot tio s. ot t s t o s i ci t it rt v E v). rt r or t ri l s

$$\delta^i_{uv} \quad \begin{cases} \pi_i \ u) < \pi_i \ v) \\ \text{ot r is ,} \end{cases}$$

c r ct ri t p r t tio π_i o t it l r. δ^i_{uv} q ls i o l i rt u is pl c or rt v i t or ri π_i. or c p ir o o s u, v V_i it u v t crossi r c_{uv} is to t r o crossi s t E u) E v) i π_i $u) < \pi_i$ v) ol s. si t s ot tio s or i p r t tio π_i t r o crossi s t l rs $i -$ i $i +$ is

$$C \ \pi_i) \quad \sum_{u \ V_\cdot} \sum_{v \ V_\cdot} \delta^i_{uv} c_{uv}.$$

or r to sp p t co p t tio o t r o crossi s i c st p i tro c t r i sio l crossi tri $c[i, p \ , p \ ]$ ic o l r q ir s p t op r tio s t r i t rc i t o rtic s. tr $c[i, p \ , p \ ]$ is s t to $c_{\pi^-_\cdot \ p_\cdot \ , \pi^-_\cdot \ p_\cdot}$ it r sp ct to l r i it co t i s t r o crossi s t t s i ci t to rt π^-_i p) it s i ci t to rt π^-_i p) i p $< p$ ol s.

o ss t t s p t o rtic s u v o l r i it p π_i u) p π_i v). s ll tri s o t crossi tri or l r i col p to i t rc it t tri s ro col p . lo o sl ll tri s o ro p o l r i to i t rc it ro p o l r i. ot t t o l op r t o t crossi tri o ot to co si r t r p its l . is s to c ll c ti t o rtic s r s pp .

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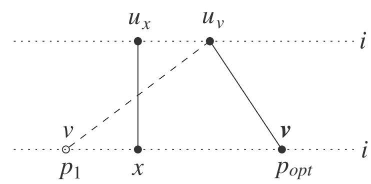

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$$LB \quad v) \quad \sum^{V} \sum^{V} \delta^i_{u\,u} c_{u\,u} + \sum_{u}^{V} \delta^i_{uv} c_{uv} + \sum_{u}^{V} \delta^i_{vu} \quad i \quad c_{vu}, c_{uv}$$

$$C\ \pi_i) - \sum_{u}^{V} \delta^i_{vu} \quad , c_{vu} - c_{uv}$$

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Lower Bounds for the Number of Bends in Three-Dimensional Orthogonal Graph Drawings

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Abstract. In this paper we present the first non-trivial lower bounds for the total number of bends in 3-D orthogonal drawings of maximum degree six graphs. In particular, we prove lower bounds for the number of bends in 3-D orthogonal drawings of complete simple graphs and multi-graphs, which are tight in most cases. These result are used as the basis for the construction of infinite classes of c-connected simple graphs and multigraphs ($2 \leq c \leq 6$) of maximum degree Δ ($3 \leq \Delta \leq 6$) with lower bounds on the total number of bends for all members of the class. We also present lower bounds for the number of bends in general position 3-D orthogonal graph drawings. These results have significant ramifications for the '2-bends' problem, which is one of the most important open problems in the field.

1 Introduction

The *3-D orthogonal grid* consists of *grid-points* in 3-space with integer coordinates, together with the axis-parallel *grid-lines* determined by these points. A *3-D orthogonal drawing* of a graph places the vertices at grid-points and routes the edges along sequences of contiguous segments of grid-lines. Edges are allowed to contain bends and can only intersect at a common vertex. 3-D orthogonal drawings have been studied in [2,3,4,5,6,7,8,12,13,14]. For brevity we say a 3-D orthogonal graph drawing is a *drawing*. A drawing with no more than b bends per edge is called a b-*bend drawing*. The graph-theoretic terms 'vertex' and 'edge' also refer to their representation in a drawing. The *ports* at a vertex v are the six directions, denoted by $\{X^+, X^-, Y^+, Y^-, Z^+, Z^-\}$, which the edges incident with v can use. For each dimension $I \in \{X, Y, Z\}$, the I^+ (respectively, I^-) port at a vertex v is said to be *extremal* if v has maximum (minimum) I-coordinate taken over all vertices. Clearly, orthogonal drawings can only exist for graphs with maximum degree six.

Drawings with many bends appear cluttered and are difficult to visualise. Therefore minimising the number of bends, along with minimising the bounding box volume, have been the most commonly proposed aesthetic criteria for

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measuring the quality of a drawing. Using straightforward extensions of the corresponding 2-D NP-hardness results, optimising each of these criteria is NP-hard [4]. Kolmogorov and Barzdin [6] established a lower bound of $\Omega(n^{3/2})$ for the bounding box volume of a drawings of n-vertex graphs. In this paper we establish the first non-trivial lower bounds for the number of bends in 3-D orthogonal drawings. Lower bounds for the number of bends in 2-D orthogonal graph drawings have been established by Tamassia *et al.* [9] and Biedl [1].

Lower bounds for the maximum number of bends per edge: Obviously every drawing of K_3 has at least one bend. It follows from results in multi-dimensional orthogonal graph drawing by Wood [11] that every drawing of K_5 has an edge with at least two bends. It is well known that every drawing of $6K_2$ has an edge with at least three bends, and easily seen that $2K_2$ and $3K_2$ have at least one edge with at least one and two bends, respectively. Here kK_2 is the 2-vertex multigraph with k edges.

A natural candidate for the existence of a simple graph with a 3-bend edge in every drawing is K_7, as originally conjectured by Eades *et al.* [5]. A counterexample to this conjecture, namely a drawing of K_7 with at most two bends per edge, was first exhibited by Wood [11]. A more symmetric drawing of K_7 with at most two bends per edge is shown in Fig. 1(a). This drawing has the interesting feature of rotational symmetry about the line $X = Y = Z$. One may consider the other 6-regular complete multi-partite graphs $K_{6,6}$, $K_{3,3,3}$ and $K_{2,2,2,2}$ to be potential examples of simple graphs with a 3-bend edge in every drawing. 2-bend drawings of these graphs are presented in [14].

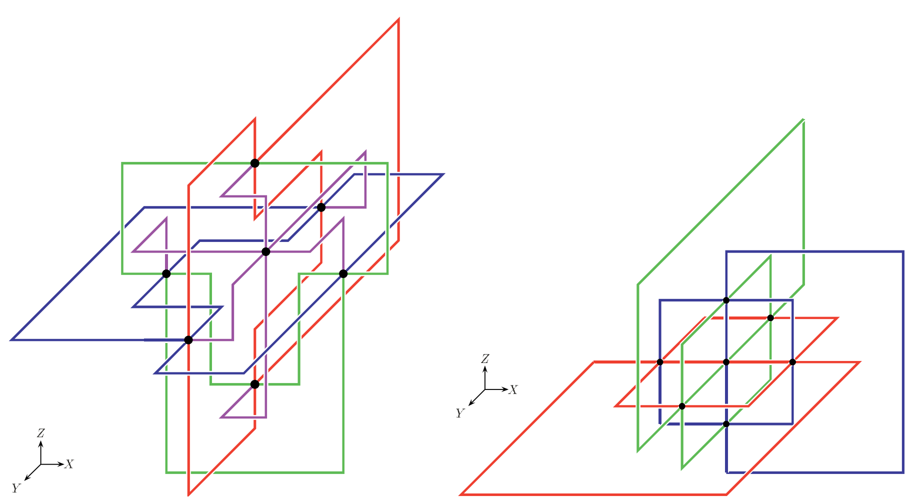

Fig. 1. (a) 2-bend drawing of K_7 (b) 4-bend drawing of K_7 with 24 bends.

Lower bounds for the total number of bends: In this paper we prove that drawings of the complete graphs K_4, K_5, K_6 and K_7 have at least 3, 7, 12 and 20 bends, respectively. For each of these graphs except K_7 there are well-

known drawings with the corresponding number of bends. Figure 1(b) shows a drawing of K_7 with a total of 24 bends (compared with the total of 42 bends for the 2-bend drawing). We conjecture that there is no drawing of K_7 with fewer than 24 bends.

We use these lower bounds for the number of bends in complete graphs as the basis for the construction of infinite families of c-connected graphs of maximum degree Δ with lower bounds on the number of bends for each member of the class. Table 1 summarises these lower bounds.

Table 1. Lower bounds for the number of bends in drawings of m-edge c-connected graphs with maximum degree Δ.

Connectivity c	Simple Graphs				Multigraphs			
	$\Delta = 6$	$\Delta = 5$	$\Delta = 4$	$\Delta = 3$	$\Delta = 6$	$\Delta = 5$	$\Delta = 4$	$\Delta = 3$
0	$\frac{20}{21}m$	$\frac{4}{5}m$	$\frac{7}{10}m$	$\frac{1}{2}m$	$2m$	$\frac{8}{5}m$	$\frac{3}{2}m$	$\frac{4}{3}m$
2	$\frac{3}{4}m$	$\frac{7}{11}m$	$\frac{3}{7}m$	$\frac{1}{4}m$	$\frac{4}{3}m$	$\frac{6}{5}m$	m	$\frac{2}{3}m$
3	$\frac{8}{11}m$	$\frac{14}{23}m$	$\frac{2}{5}m$	$\frac{2}{9}m$	m	$\frac{4}{5}m$	$\frac{1}{2}m$	-
4	$\frac{12}{17}m$	$\frac{7}{12}m$	$\frac{3}{8}m$	-	$\frac{2}{3}m$	$\frac{2}{5}m$	-	-
5	$\frac{24}{35}m$	$\frac{14}{25}m$	-	-	$\frac{1}{3}m$	-	-	-
6	$\frac{2}{3}m$	-	-	-	-	-	-	-

Upper bounds: A number of algorithms have been proposed for 3-D orthogonal graph drawing [2,3,5,6,8,12,14] which explore the apparent tradeoff between the maximum number of bends per edge and the bounding box volume (see [14] for an overview). We now summarise the known upper bounds on the number of bends in the drawings produced by these algorithms. The 3-Bends algorithm of Eades *et al.* [5] and the Incremental algorithm of Papakostas and Tollis [7] both produce 3-bend drawings[1] of multigraphs[2] with maximum degree six. As discussed above there exists simple graphs with at least one edge having at least two bends in every drawing. The following problem is therefore of considerable interest:

2-Bends Problem: Does every simple graph with maximum degree six admit a 2-bend drawing? [5]

[1] The 3-Bends algorithm [5] produces drawings with $27n^3$ volume. By deleting grid-planes not containing a vertex or a bend the volume is reduced to $8n^3$. The Incremental algorithm [7] produces drawings with $4.63n^3$ volume. A modification of the 3-Bends algorithm by Wood [14] produces drawings with $n^3 + o(n^3)$ volume.

[2] The 3-Bends algorithm [5] explicitly works for multigraphs. The Incremental algorithm, as stated in [7], only works for simple graphs; with a suitable modification it also works for multigraphs [A. Papakostas, private communication, 1998].

The Diagonal Layout and Movement algorithm of Wood [12] solves the 2-bends problem in the affirmative for simple graphs with maximum degree five. For maximum degree six simple graphs, the same algorithm uses a total of at most $7m/3$ bends, which is the best known upper bound for the total number of bends in 3-D orthogonal drawings.

In this paper we provide a negative result related to the 2-bends problem. A 3-D orthogonal graph drawing is said to be in *general position* if no two vertices lie in a common grid-plane. The general position model is used in the 3-Bends and Diagonal Layout and Movement algorithms. In this paper we show that the general position model, and the natural variation of this model where pairs of vertices share a common plane, cannot be used to solve the 2-bends problem, at least for 2-connected graphs.

The remainder of this paper is organised as follows. In Sect. 2 we establish a number of introductory results concerning 0-bend drawings of cycles. These results are used to prove our lower bounds on the total number of bends in drawings of complete graphs, established in Sect. 3. In Sect. 4 we use these lower bounds as the basis for lower bounds on the number of bends in infinite families of graphs of varying connectivity and maximum degree. In Sect. 5 we present lower bounds for the number of bends in general position drawings and their implications for the 2-bends problem.

Throughout this paper we consider n-vertex m-edge loop-free graphs with maximum degree six. By C_m we denote the m-edge cycle. A *chord* of a cycle C is an edge not in C whose end-vertices are both in C. We say two cycles are *chord-disjoint* if they do not have a chord in common. In this paper most proofs are outlined, and the proofs of the results in Table 1 concerning multigraphs are omitted; see [13] for details of all the proofs.

2 Drawings of Cycles

In this section we characterise the 0-bend drawings of the cycles C_k ($k \leq 7$). We then show that if a drawing of a complete graph contains such a 0-bend drawing of a cycle then there must be many edges with at least three bends in the drawing of the complete graph. These results are used in Sect. 3 in our lower bounds for the total number of bends in drawings of complete graphs.

A straight-line path in a 0-bend drawing of a cycle is called a *side*. A side parallel to the I-axis for some $I \in \{X, Y, Z\}$ is called an I-side, and I is called the *dimension* of the side. Clearly the dimension of adjacent sides is different, thus in a 2-dimensional drawing the dimension of the sides alternate around the cycle. Hence, there is no 2-dimensional 0-bend drawing of a cycle with an odd number of sides. If there is an I-side in a drawing of a cycle for some $I \in \{X, Y, Z\}$ then clearly there is at least two I-sides. Therefore a drawing of a cycle with X-, Y- and Z-sides, which we call *truly 3-dimensional*, must have at least six sides. Hence there is no truly 3-dimensional 3-, 4- or 5-sided 0-bend drawing of a cycle. There is also no two-dimensional 3- or 5-sided 0-bend drawing of a cycle. We therefore have the following observation.

Observation 1 *(a) There is no 3- or 5-sided 0-bend drawing of a cycle,*
(b) there is no 0-bend drawing of C_3*, and*
(c) all 0-bend drawings of C_4 *and* C_5 *have four sides.* □

The next result forms an important component of the lower bounds to follow.

Lemma 1. *If a drawing of a complete graph contains a 0-bend 4-cycle (respectively, 5-cycle) then there are at least two (four) chords of the cycle each with at least three bends.*

Proof. By Obs. 1(c) all 0-bend drawings of C_4 and of C_5 have four sides. As illustrated in Fig. 2(a), the chord connecting each pair of diagonally opposite vertices in a 4-sided drawing of a cycle has at least three bends. Hence, if a drawing of a complete graph contains a 0-bend C_4, then the two chords each have at least three bends. Also, in the case of C_5, the two edges from the vertex not at the intersection of two sides to the diagonally opposite vertices both have at least three bends, as in Fig. 2(b). Hence, if a drawing of a complete graph contains a 0-bend C_5, then the four chords each have at least three bends. □

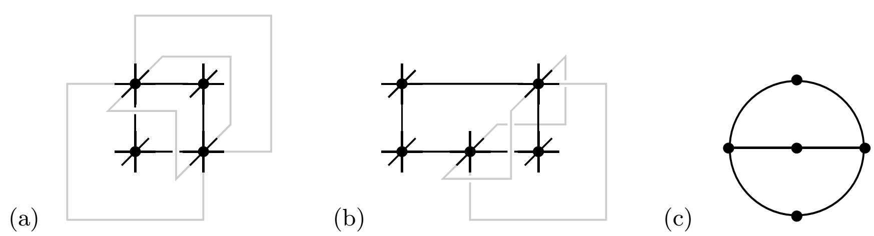

Fig. 2. 3-bend edges 'across' (a) C_4 and (b) C_5. (c) The graph H_1

Consider the graph shown in Fig. 2(c), which we call H_1.

Observation 2 H_1 *does not have a 0-bend drawing.*

Proof. H_1 contains C_4. As in Lemma 1, an edge between the non-adjacent vertices of a 4-sided cycle needs at least three bends. Hence the 2-path in H_1 between the non-adjacent vertices of the 4-cycle has at least one bend. Therefore H_1 does not have a 0-bend drawing. □

We now consider 6-sided 0-bend drawings of a cycle.

Lemma 2. *If a drawing of a complete graph contains a 0-bend 6-cycle then there are at least four chords of the cycle each with at least three bends.*

Proof. We can assume without loss of generality that there is a 0-bend drawing of C_6 contained in a drawing of K_6. By Obs. 1(a), all 0-bend drawings of C_6 have four or six sides. In a 4-sided 0-bend drawing of C_6 the two vertices not at the intersection of adjacent sides can be (a) on the same side, (b) on adjacent sides, or (c) on opposite sides. In each case there are at least six chords from a

vertex at the intersection of two sides to a vertex on neither of these sides, each with at least three bends.

Case analysis shows that the only 6-sided 0-bend drawings of C_6 (up to symmetry) are those in Fig. 3. For each such drawing, the chords of C_6 shown in Fig. 3 each require at least three bends. In the case of the drawing in Fig. 3(c) there are at least six chords each requiring at least three bends. For the drawing in Fig. 3(a) (respectively, Fig. 3(b)) it can be shown, by considering certain sets of chords for which all edge routes with at most two bends pass through a single grid point, that a further two (four) chords have at least three bends. Hence there are at least six chords of the cycle each with at least three bends. □

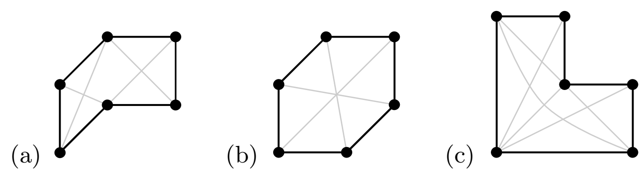

Fig. 3. Edges with at least 3 bends in a drawing of K_6 containing a 6-sided 0-bend C_6.

We now consider 7-sided 0-bend drawings of a cycle.

Lemma 3. *If a drawing of K_7 contains a 0-bend 7-cycle then there are at least four chords of the cycle each with at least three bends.*

Proof. By Obs. 1(a), a 0-bend drawing of C_7 has four, six or seven sides. In a 4-sided 0-bend drawing of C_7 the three vertices not at the intersection of two adjacent sides can be (a) all on the same side, (b) two on one side and one on an adjacent side, (c) two on one side and one on the opposite side, or (d) all on different sides. In each case there are at least eight chords from a vertex at the intersection of two sides to a vertex on neither of these sides, each with at least three bends. The 6-sided 0-bend drawings of C_7 can be obtained from the 6-sided 0-bend drawings of C_6 by placing one new vertex at each of the essentially different edges of each drawing. By Lemma 2, at least six of the chords of C_6, and therefore of C_7, have at least three bends. Case analysis shows that the only 7-sided 0-bend drawings of C_7 are those shown in Fig. 4. For each drawing there are at least four chords which need at least three bends. □

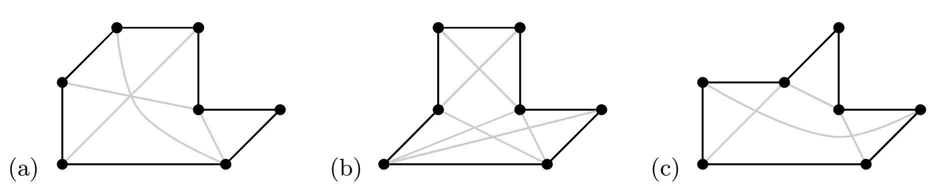

Fig. 4. Edges with at least three bends in a 7-sided 0-bend drawing of C_7.

3 Drawings of Complete Graphs

In this section we establish lower bounds for the total number of bends in 3-D orthogonal drawings of K_4, K_5, K_6 and K_7. We omit our proofs that the well-known drawings of K_4, K_5 and K_6 shown in Fig. 5 are bend-minimum. They are similar to the proof of the lower bound for K_7 which follows.

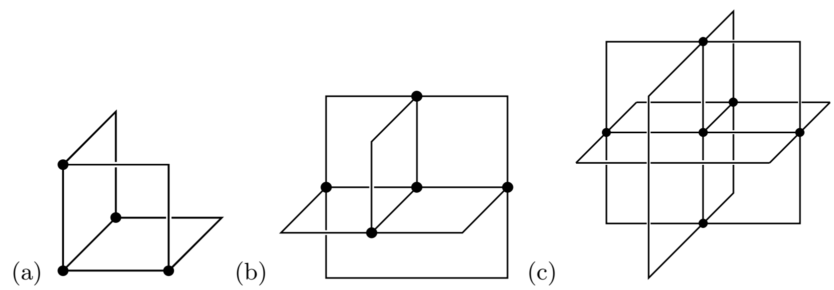

Fig. 5. (a) K_4 with 3 bends, (b) K_5 with 7 bends and (c) K_6 with 12 bends.

Theorem 1. *Every drawing of K_4 has at least three bends. Every drawing of K_5 has at least seven bends. Every drawing of K_6 has at least twelve bends.* □

Figure 1(b) shows a drawing of K_7 with a total of 24 bends.

Theorem 2. *Every drawing of K_7 has at least 20 bends.*

Proof. Suppose to the contrary, that there is a drawing of K_7 with at most 19 bends. The subgraph of K_7 consisting of the 0-bend edges is called the *0-bend subgraph.* Let k_i $(i \geq 0)$ be the number of i-bend edges. Hence

$$\sum_{i\geq 0} k_i \;=\; 21, \text{ and } \sum_{i\geq 1} ik_i \;\leq\; 19 \;. \tag{1}$$

Case analysis shows that every subgraph of K_7 with at least ten edges contains C_3 or H_1. By Obs. 1(b) and Obs. 2, the graphs C_3 and H_1 do not have 0-bend drawings. Hence $k_0 \leq 9$. Suppose $k_0 = 8$ or $k_0 = 9$. By (1)

$$12 \;\leq\; \sum_{i\geq 1} k_i \;\leq\; 19 - \sum_{i\geq 1}(i-1)k_i$$
$$\sum_{i\geq 2}(i-1)k_i \;\leq\; 7 \;. \tag{2}$$

Case analysis shows that every subgraph of K_7 with at least eight edges contains a cycle C_k $(k \neq 4)$, two chord-disjoint cycles, or an H_1 subgraph. Therefore the 0-bend subgraph contains a cycle C_k $(k \geq 5)$ or two chord-disjoint subgraphs (since C_3 and H_1 do not have 0-bend drawings by Obs. 1(b) and Obs. 2, respectively). If the 0-bend subgraph contains a cycle C_k $(k \geq 5)$ then by Lemma 1, Lemma 2 and Lemma 3 there are at least four chords of the

cycle each with at least three bends. On the other hand if the 0-bend subgraph contains two chord-disjoint cycles then these cycles have length at least four; thus by Lemma 1, Lemma 2 and Lemma 3, two chords from each of these cycles each have at least three bends. In either case the drawing of K_7 has at least four edges each with at least three bends; that is,

$$\begin{aligned} 4 &\leq \sum_{i\geq 3} k_i \\ k_2 + 8 &\leq k_2 + \sum_{i\geq 3} 2k_i \leq \sum_{i\geq 2}(i-1)k_i \\ k_2 + 8 &\leq 7 \qquad\qquad\qquad\qquad \text{(by (2))} \end{aligned}$$

Hence $k_2 \leq -1$, which is a contradiction. Therefore $k_0 \leq 7$. By (1) with $k_0 \leq 7$

$$\begin{aligned} 14 \leq \sum_{i\geq 1} k_i &\leq 19 - \sum_{i\geq 1}(i-1)k_i \\ \sum_{i\geq 2}(i-1)k_i &\leq 5 \\ k_2 + 2\sum_{i\geq 3} k_i &\leq 5 \end{aligned} \tag{3}$$

Let A be the set of edges of K_7 routed using an extremal port at exactly one end-vertex. Let B be the set of edges routed using extremal ports in the same direction at its end-vertices. Let C be the set of edges routed using extremal ports in differing directions at its end-vertices. K_7 is 6-regular, thus all ports are used. Since there is at least one extremal port in each direction, we have $|A| + |B| + 2|C| \geq 6$. It is easily seen that an edge in A or B has at least two bends, and that an edge in C has at least three bends. Hence

$$k_2 + 2\sum_{i\geq 3} k_i \geq 6 \ .$$

This contradicts (3). The result follows. □

4 Constructing Large Graphs

In this section we use the lower bounds for the total number of bends in drawings of the complete graphs established in Sect. 3 as building blocks to construct infinite families of graph with lower bounds for the number of bends.

Given graphs G and H with $|V(G)| \geq \Delta(H)$, we define $H\langle G\rangle$ to be the graph obtained by replacing each vertex of H by a copy of G, and connecting the vertices adjacent to a vertex $v \in V(H)$ to different vertices in the copy of G corresponding to v. In most cases, H is regular and G is a complete graph, thus $H\langle G\rangle$ is well-defined. In other cases we shall specify the mapping between edges incident to v and the vertices in the copy of G corresponding to v. We

also employ the *cartesian product* $G \times H$ of graphs G and H to construct larger graphs. $G \times H$ has vertex set $V(H) \times V(G)$ with (v_1, w_1) and (v_2, w_2) adjacent in $G \times H$ if either $v_1 = v_2$ and $w_1 w_2 \in E(H)$, or $w_1 = w_2$ and $v_1 v_2 \in E(G)$.

By taking disjoint copies of K_7, K_6, K_5 and K_4 the next result follows immediately from Theorem 1 and Theorem 2.

Theorem 3. *There exists infinite families of simple (disconnected) m-edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{20}{21}m$ ($\frac{4}{5}m$, $\frac{7}{10}m$ and $\frac{1}{2}m$) bends in every drawing.* □

To obtain our lower bounds for 2-, 3- and 4-connected graphs consider the graphs $C_r\langle K_p\rangle$ $(p \geq 3)$, $(C_r \times K_2)\langle K_p\rangle$ $(p \geq 3)$ and $(C_r \times C_3)\langle K_p\rangle$ $(p \geq 4)$ for some $r \geq 3$, as illustrated in Fig. 6.

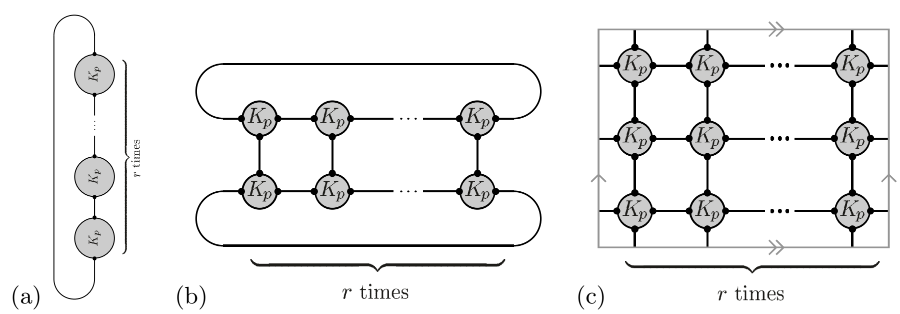

Fig. 6. (a) 2-connected $C_r\langle K_p\rangle$, (b) 3-connected $(C_r \times K_2)\langle K_p\rangle$, (c) 4-connected $(C_r \times C_3)\langle K_p\rangle$.

Theorem 4. *There exists infinite families of simple 2-connected m-edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{3}{4}m$ ($\frac{7}{11}m$, $\frac{3}{7}m$ and $\frac{1}{4}m$) bends in every drawing.*

Proof. The graphs $C_r\langle K_6\rangle$ with $r \geq 2$ have maximum degree six and $m = 16r$ edges. By Theorem 1, K_6 has at least 12 bends in every drawing, thus $C_r\langle K_6\rangle$ has at least $12r = \frac{3}{4}m$ bends in every drawing. The graphs $C_r\langle K_5\rangle$ with $r \geq 2$ have maximum degree five and $m = 11r$ edges. By Theorem 1, K_5 has at least 7 bends in every drawing, thus $C_r\langle K_5\rangle$ has at least $7r = \frac{7}{11}m$ bends in every drawing. The graphs $C_r\langle K_4\rangle$ with $r \geq 2$ have maximum degree four and $m = 7r$ edges. By Theorem 1, K_4 has at least 3 bends in every drawing, thus $C_r\langle K_4\rangle$ has at least $3r = \frac{3}{7}m$ bends in every drawing. The graphs $C_r\langle K_3\rangle$ with $r \geq 2$ have maximum degree three and $m = 4r$ edges. By Obs. 1(b), K_3 has at least 1 bend in every drawing, thus $C_r\langle K_3\rangle$ has at least $r = \frac{1}{4}m$ bends in every drawing. Clearly $C_r\langle K_p\rangle$ is 2-connected. □

The proofs of the following results for 3- and 4-connected graphs are similar to the proof of Theorem 4.

Theorem 5. *There exists infinite families of simple 3-connected m-edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{8}{11}m$ ($\frac{14}{23}m$, $\frac{2}{5}m$ and $\frac{2}{9}m$) bends in every drawing.* □

Theorem 6. *There exists an infinite family of simple 4-connected m-edge graphs with maximum degree six (respectively, five and four) with at least $\frac{12}{17}m$ ($\frac{7}{12}m$ and $\frac{3}{8}m$) bends in every drawing.* □

To obtain our lower bounds for 5- and 6-connected graphs consider the graphs $(C_r \times C_3 \times K_2)\langle K_p\rangle$ ($p \geq 5$) and $(C_r \times C_3 \times C_3)\langle K_6\rangle$ for some $r \geq 3$, as illustrated in Fig. 7. Again the proofs are very similar to that of Theorem 4.

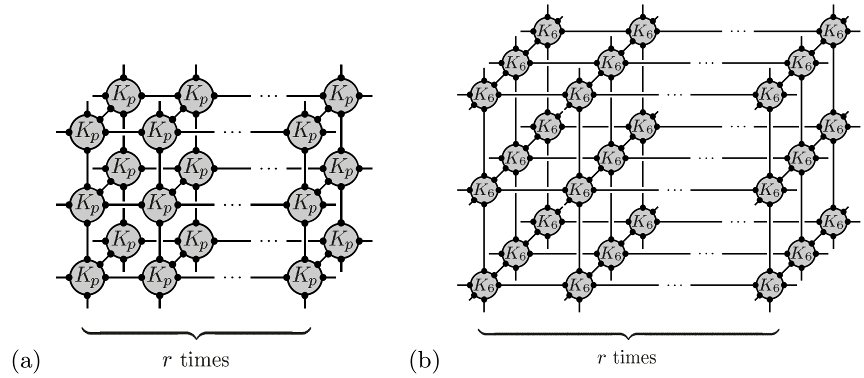

Fig. 7. (a) 5-connected $(C_r \times C_3 \times K_2)\langle K_p\rangle$, (b) 6-connected $(C_r \times C_3 \times C_3)\langle K_6\rangle$.

Theorem 7. *There exists infinite families of simple 5-connected m-edge graphs with maximum degree six (respectively, five) with at least $\frac{24}{35}m$ ($\frac{14}{25}m$) bends in every drawing.* □

Theorem 8. *There exists an infinite family of simple 6-connected m-edge graphs with maximum degree six with at least $\frac{2}{3}m$ bends in every drawing.* □

5 General Position Drawings and the 2-Bends Problem

Recall that a 3-D orthogonal graph drawing is in general position if no two vertices lie in a common grid-plane. In this section we establish lower bounds for the number of bends in general position drawings.

Lemma 4. *If the graph G has at least k bends in every general position drawing then for any edge e of G the graph $G \setminus e$ has at least $k-4$ bends in every general position drawing.*

Proof. If $G \setminus e$ has a drawing with b bends then, the edge e can be inserted into the drawing with at most four bends and the edges rerouted so that there are no edge crossings [12,14]. Thus, there is a general position drawing of G with $b+4$ bends. Every general position drawing of G has at least k bends, hence $b+4 \geq k$ and $b \geq k-4$. □

Clearly every edge in a general position drawing has at least two bends. The following lower bounds for general position drawings are based on the observation that an edge routed using an extremal port in a general position drawing has at least three bends. For 6-regular m-edge graphs all ports must be used, thus such a graph has at least $2m + 6$ bends in every general position drawing. Hence the graphs consisting of disjoint copies of K_7 provide the following lower bound.

Lemma 5. *There exists an infinite family of n-vertex m-edge simple graphs, each with at least* $2m + 6n/7$ *bends in every general position drawing.* □

Note that for 6-regular graphs the above lower bound is within $m/21$ of the upper bound of $7m/3$ for the total number of bends in general position drawings established by the DIAGONAL LAYOUT AND MOVEMENT algorithm [12].

To establish our lower bound for general position drawings of 2-connected graphs consider the graph $C_r\langle K_7 \setminus e\rangle$ for $r \geq 2$, where the non-adjacent vertices of each $K_7 \setminus e$ are incident to the edges of C_r, as illustrated in Fig. 8(a).

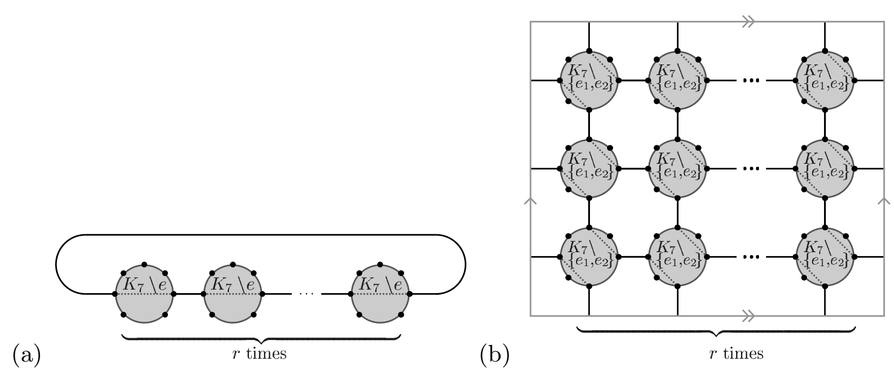

Fig. 8. (a) 2-connected $C_r\langle K_7 \setminus e\rangle$, (b) 4-connected $(C_r \times C_3)\langle K_7 \setminus \{e_1, e_2\}\rangle$.

Lemma 6. *There is an infinite family of n-vertex m-edge simple 2-connected graphs, each with at least* $2m + 4n/7$ *bends in every general position drawing.*

Proof. Clearly $C_r\langle K_7 \setminus e\rangle$ is 2-connected. K_7 has at least $2|E(K_7)| + 6$ bends in every general position drawing. Thus by Lemma 4, a general position drawing of $K_7 \setminus e$ has at least $2|E(K_7)| + 6 - 4 = 2|E(K_7 \setminus e)| + 4$ bends. The edges of C_r each have at least two bends, thus $C_r\langle K_7 \setminus e\rangle$ has at least $2m + 4n/7$ bends in every general position drawing. □

To establish our lower bound for general position drawings of 4-connected graphs consider the graph $(C_r \times C_3)\langle K_7 \setminus \{e_1, e_2\}\rangle$ for $r \geq 2$, where for each copy of $K_7 \setminus \{e_1, e_2\}$, e_1 and e_2 are edges of K_7 with no common end-vertices, and these end-vertices are incident to different edges in $C_r \times C_3$, as illustrated in Fig. 8(b).

The proof of the next result is very similar to that of Lemma 6.

Lemma 7. *There is an infinite family of n-vertex m-edge simple 4-connected graphs, each with at least* $2m + 2n/7$ *bends in every general position drawing.* □

We now look at the ramifications of the above general position lower bounds for the 2-bends problem. Edges with at most two bends can be classified as 0-bend, 1-bend, 2-bend planar or 2-bend non-planar. As illustrated in Fig. 9, a given 2-bend drawing can be transformed into a general position drawing whose number of bends depends on the number of 0-bend and 2-bend planar edge routes in the 2-bend drawing. We omit the proof.

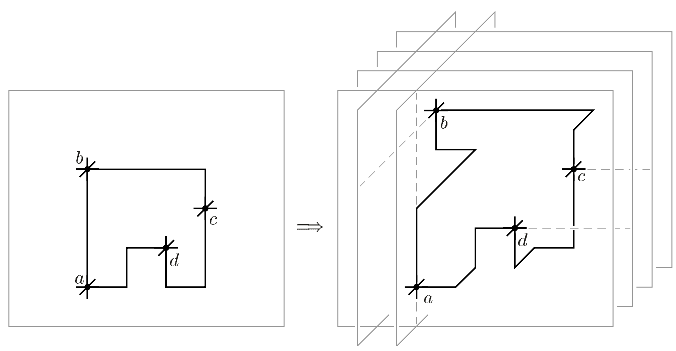

Fig. 9. Removing a plane containing many vertices.

Lemma 8. *If in a 2-bend drawing of an m-edge graph G the number of 0-bend edges is k_0 and the number of 2-bend planar edges is k_2', then there exists a general position 3-D orthogonal drawing of G with $2m + k_0 + k_2'$ bends.* □

Corollary 1. *There exists an infinite family of 6-regular n-vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k_2' \geq 6n/7$.*

Proof. By Lemma 5, there exists an infinite family of graphs, each with at least $2m + 6n/7$ bends in any general position drawing. If there is a 2-bend drawing of such a graph, then by Lemma 8 there is a general position drawing with $2m + k_0 + k_2'$ bends. Hence $2m + k_0 + k_2' \geq 2m + 6n/7$ and $k_0 + k_2' \geq 6n/7$. □

The following two results are obtained using the same argument used in the proof of Corollary 1 applied with Lemma 6 and Lemma 7, respectively.

Corollary 2. *There exists an infinite family of 6-regular 2-connected n-vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k_2' \geq 4n/7$.* □

Corollary 3. *There exists an infinite family of 6-regular 4-connected n-vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k_2' \geq 2n/7$.* □

A natural variation of the general position model allows at most two vertices in any one grid-plane with each vertex being coplanar with at most one other vertex. In this model there is at most $n/2$ pairs of coplanar vertices and hence at most $n/2$ planar edge routes. Since $n/2 \leq 4n/7$, it follows from Corollary 2 that there exists graphs which do not have 2-bend drawings in this model.

Theorem 9. *There exists an infinite family of 2-connected graphs each of which does not have a 2-bend drawing with at most two vertices in any one grid-plane and with each vertex being coplanar with at most one other vertex.* □

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3 ⋆

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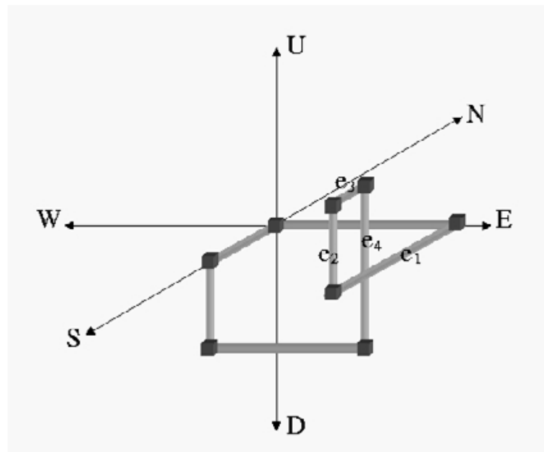

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o ti i t l b l o o i l q b i t ot-
tio . or l i t rib o i l q o i-
r tio l b l r . t i ot tio t or r o t l b l i
ot i i i rit ro t o rti l r b q
b o .
o ti i ti i t l b l i o i l q ro t ot r
l b l o it i l ot tio . o t t i o i l q
ot o l t t t o i l q o t i
ir tio l b l b t l o t t t l b l i i i l t o t
l b l i o o .
t t t l t o t t o r i o i l q r
. t i l to r ll t t t i q t t
l i ir l r q . o i l q or t
i q o i l q or l i ir l r q .

o k t o tr ti roo t t l t t o t i
o i l q o l t i it i l ort o o l r i .

i t itio b i o r o tr tio o r i or l i to i i
t t it ill b l bor tio o l o i to b r lo t
o bo . l o l t i o o t o ti ll i r t
l i o r fl t or
i i fl t . ro o i l q o l t i
obt i (o ibl t r o o i tio) t i l b l o ir or

k to ollo t o bi bo it t r i i l b l to b r t o ort lo t r or r o t bi bo r i to o t to t r t i lo . rl i ir or k o i tl llo to l t o ti t o ort t i iti t o t t b i i lo l t to t o i l l b l . r ti t r i ro o ot ril i lo l t to t o i l l b l b t t i t l o l i t b i i or t o tr tio .

i lt i t t r o ot o t i ir o o o it l ir t l b l t t r t o i l q o o t i ir o o o it l ir t l b l t t r t l t o . o i l q o l t ot ro i t o i t rl i ir or k or t o tr tio . i oti t t ollo i itio .

tro o i l l *t t t t t t t t*

t l r ol t i lt . t roo i t i l l i i i oi o o i l l b l r b tit t or ol o .

t t t t t t

i tro o i l q o l t i (i b o i li r ti i o i t) o t i l r i or t o ti t b t t o i l l b l t i l t to t o i l l b l o t t t r i r i i r t o t t (t i i o ibl b t rl i ir or k or o t o i l l). o r t t t i i t to r t t t l lo or l t ol t o li r i q liti r i t o tr i t .

i b tio rib o to o tr t r i ro tro o i l q .

r i $\Gamma($ $)$ o t i i t tr l o it rt r i it i ir tio t t tr oi t it r t to t t ir tio o t r io t o $\Gamma($ $)$. r i $\Gamma($ $)$ o t i *t* i it r t l t b r l b rbitr ril lo it o t r ti i t r tio it i t r i o t t t .

t t t t
t t t t
t t t t t t t t t t t t fl t t
t t t t ( ) t

bri fl r i t roo o t il r b lo or o r l o tr tio it roo o orr t . r t rt o t l ollo ro t l orit i t r o t itio o i r i . or t o rt ot t t i o i t o tl t o l b l t it i l rl o bl t ibl .

o o t l t t o fl t . b q tri tl b t t r t l t l t t o t i tr itio l b l. l t ti o t r t tr itio l b l t t ori i . orki t ro ro t i r t tr itio l b l r t i r i or t i iti l b q o . t tr itio l b l i t r t l b l to b r l t . t ll t r t l b l o i r it b r rbitr ril lo .

o r t r i r o o i r t l b l t ti i t l ollo t r t tr itio l b l. t t b r it it t il t t ori i r i- l r to t l o t r io fl t. orki ro t i l b l r t i r i i t r l t t l b l i r it t rt r b i it ir tio t r io l r t t t o t t o l b rbitr ril lo . t r t t t t tr itio l b l i l o i l t t ll t l t t o i r it b rbitr ril lo . i o l t r i o t roo .

ot t t i t bo roo k t t ti o t t-to-l t t o $\Gamma($ $)$ t t il o it l t t li o t bo i bo o t r i o t r i i i t r l l b l o .

o rib o to obt i r i or l it o i l b q . i or it t t o t o o it l ir t l b l . o l o ro t r i i t t (o b t).

o o t o ti t b k o t t o l t o t t bo it. l t i t o i t o t t t o l t o it t o t i t l t t o fl t . t r i t t o l t o i r ot t i o l li i t fl t o o tr i ti t t t t i o i l.

o ti rt o t o bl t ibl r i ill b l i r t o t t . t o r r i l t t o t i o bl t ibl r i b r rbitr ril lo . ir l t ill b o o lo t t t o t t i t r l rt o t o bl t ibl r i i ol t i i ti t o t t .

k t r i (t ir bo i bo) or t i o ti t bo . o o t b t oi t . l ti to lo l ori i o r i k o t oor i t o ll oi t o t t i t t rt o t r i .

o t r i l t or t o i l t o itio t ori i o t bo i bo .

ook t t o . i it i tro o i l q t r r t t o o ibiliti t ir (. .) or t k (. .). t i to t r i o t t or t l t o t bo i

bo o t i i t o ti r i . ot t t o t o bo r i to t o t t.

t . . ot t k o l t to b i to t o i l t . i l t o q tio i q liti t b ti b t k o or o o it l ir t ir o o i l t t tot l l t o ll t ir t t q l t tot l l t o ll t ir t i il rl or t ot r ir . i ill r t t t t l lo .

ot t t lr t r i t l t o t t t t r ot o i l ll t lo tio o t oi t o t o i l t i t r o t lo l oor i t o t bo . it to t r i or lo l ori i o bo t o t r i q liti o or o t t r ort o o l ir tio t t r t t t t bo t tri tl i i it i o t t.

ti i t t o i q liti i li t t orr o i t o i q liti o t l t o t o i l t t l o b ti . i i lo r bo o t l t o o i l t o t or . . or o o t t . o o .

o r t t t l ill lo to t t o i q liti o l t t r q tio o or ir o o o it ir tio ollo . tot l l t o t ir t t q l t tot l l t o t ir t i il rl or t ot r t o ir . or or t q tio i it r . . + . . or . . + . . or o o iti o t t . . i il rl or t ot r t o ir .

o i r t o tr i t o t l t o rti l r o o it l ir t ir o . . . o tr i t r

or o - ti o t t . . or . . . . + . .
(or i . . t . . + . .
. .
. . .

b ti b i i t l

. (. . − . .) (or . (. . − . .) i . .).

i t r i t l o t l t o t o i l t ir t . r i i l t or t ot r ir tio b t r i i il rl .

l t o b o ot t t t or lo l . o t t t l i i l ot t t l rl t t t r ot o i l o ot i t r t ot r. it to k t t o o i l t i t r t ot r o i l t or o - o i l o (i l i o i bo ot lo t t t oi t o t o i l t). i ollo il ro t t t t t bo i bo or t o ti t r lo t i i ti t o t t .

o obt i li r ti l orit or t ti or t o itio o t r
or ro i o i t o i l q o l t i i li r ti .
o o t i i li r ti ir o r ll l fl t i . roo o t
it o t o itio (tio or k t) r l t ti ti t
o itio t it t o t i o o o t t b r o o i l q
o i l t b t r l tio o t l b l i t o i l q
to ot r to t t o i r ll l fl t . t o i ir l r
q t t t t t ir o r ll l fl t b o rbitr ril i
t rti l b l or l t r t l b l o o o t fl t . it i
ot r to tr l b l o t tir q t rti l
r i or o i l q o o o t i l t . o q tl
li r ti l orit b i to k or t r o o o t
i l o i l q . i tro o i l l i l ort o o l
r i or it b o tr t rib i t r io b tio .
o t tio o t oor i t o t oi t o t t o r i
r q ir ( ) ti or t r l o l o o t tio . i t l t o
o t i t r q ir ( ) bit to r or t r i ti b o
( ) or ri i o l.

i i l ort o o l r i $\Gamma($ $)$ o l o r o l i to o
t t o t i o i l q o l t i . li tl rt rbi $\Gamma($ $)$ i
r it o t lo o r lit t t $\Gamma($ $)$ ti
t t l t t o t o rti b lo i to i ti t fl t o
r r o t i - li l . l t or t t ollo
r b o t i tio .

roo i b o t i o tti $\Gamma($ $)$ i to t o t t t o
r o t t t ot r o o b k to t ori i . ollo t t o
t look or o i l q o t .
tio i tio 2 roo b o t i ro o t ollo
il ro or 2. t r q ir or l bor t i r

it bl oo t oi t r t $\Gamma($ $)$ i or r to
t t o t .. .. .
o i l q or .. ..′ .. ..′ t t o i t o t r
or o r l b l ..′ i t l b l o t r t l t l b l o .. ..′ i t
l b l o t r t l t l b l o .. .
rt i r o itio to t r it or 2 t
ro rti l o b tio .2 to o tr t o i l q
o l t i or .

o ob r o b i ro rti o o i l q o
l or t . ro rti r l to ro . l i
ot r i i t i b tio i r too to ot it r t or
l . t io o t o b q o t q
o l t r t l t o or r i .

t o t i t r l b l t t t r o ti o
t fl t o t o o it ir tio .

t t b t . r o ro it r t or l t l b l
t r lti q i till o i l or .

t 3 o i t o t r l b l t t t ll ort o o l ir -
tio t b q o i o i l q o .

t ro rt llo to r o l b l ro o i l q o
l t o r it roo i i i t o q o t itio o o
ro rt .

t o i t o o r l b l tl t o o i r o o it l ir -
t t b q obt i b l ti ro o o t t o o o it
l b l i o i l q o .

or l i i o i l q t b ro rt
i l o o i l q .
ollo i l llo to r t o o i l q to obt i
o i l q .

t *t* *t* *t*
t *t*
t *fl* *t* *t* *t* *t*

t *t* *t* *t*
t *t* *t*
t *t*
t *fl* *t* *t* *t* *t*

t *t*
t *t* *t* *t* *t*

t i it r lt or 3 t .

$t\,\Gamma(\;)$ *t* *t* *t* *t t*
t $\Gamma(\;)$ *t* *t* *t* *t*
t *t* *t* *t*

k t t roo i t t t l t l t o r fl t . tr i t-
or r l i l l it r t o r fl t .
r t r l o itio tio i t l t o r fl t t it
l t o fl t . . t t $\Gamma($.) $\Gamma($.) li o r ll l l .
t b t t rti oi t o . l t b t t rti oi t o .. t
b t r t o $\Gamma($.) l t b t r t o $\Gamma($.). t ..′ b t
ir tio l b l or l t ..′ b t ir tio l b l or . b r t t ..′
i tr itio l b l r b t o fl t o t fl t r i . fl t ..
i il rl ..′ i tr itio l b l r b t fl t r i . fl t ..
ot it ._ ._ t fl t r i . . r ti l .
b r t t i t ori i i o t t i oi t o o t t.
t o i oi t ir t t $\Gamma($.′.) i t t ro to $\Gamma($.′.)
i t t ro to . t r or t t ..′ .′. ..′ .′. . l o
.. ..′ .′. .. ..′ .′. .

o lo t t ori i t l t b t o t t o t i i
. t t r *t t* *t t* i o i t ori i
ro to l i t o t t. i il r itio b i or
t r l tio i o it r t to . b r t t i r ot
q i l t it r t to t lo t t ori i t t t
o ot li i t o t t t t o t i .

o i r o r i t r i b t r or ot r
q i l t it r t to b t r or ot r q i l t it
r t to . or o o to oo o i l q o l t
i or . i i o b i ro rti 3 3 to
r or r i o r tio o t o o i l q o i .. ..′
t ot r i .. ..′. o i l q o .. ..′ (.. ..′)
it r o i t o t r l b l i r q i l t it r t to (
r q i l t it r t to) i i or 2 i to t
o i l q or it o i t o o r l b l i r ot q i l t
it r t to (r ot q i l t it r t to) i i
6 i to t o i l q . i t .. ..′ .. ..′
r ot i oi t t ir o i l q ot b i oi t. o r t
r i o r tio r or o t o i l q or t ro
li o i l q o l t i or l .

ri t r lt o t i tio it t ollo i t or .

t $\Gamma($) *t*
t *t*

i r r t ri t o l t t it i l ort o o l
r i i 3 . r t ri tio i l li r ti r o itio l orit

r i l orit t t i li r i t r l o l ( ) i t ri i o l. t r ti r l t robl t t r i i l ( ) r t ri i i l or r t t r ot t l (2) i i i i t ol o bo i bo o l t t t b r it rti t ri oi t (t oor i t o o r r i ill b r tio l b l to b i t r o r ot tt t to i i i t ol o t r i) (3) t i t r t ri tio o t i r to l it or t i ir tio /or to i io i r t t r .

. . . l. u t - t g n l g p ng . n s s pp. 5.

. . l . m . m t n . t . t g n l 3- g p ng. s s 3( ) 63 7 .

3. . . n . . n n . u k . - m n n l g p ng. 7( ) 7.

. . tt t . . m n . ll . p ng. nt ll .

5. . tt t . tt . u n . t . m ng p l m p t t t n n t n g . n .- . u . . t ll- t . n n . m . s 6^{th} p ng - l g l. 5 pp. 6 -73 .

6. . tt t n . m . ngl pl n t ngul g p . s (3) 3 35 6.

7. . . t k n . t . t n qu m lg n n t m n n l t g n l g p ng . ss 6 7 3 6.

. . . m n n . t . - m n n l t g n l g p ng lg t m . s l. 3 pp. 55- 7 .

. . g. ult n ng ngl g p . ( ) 3 . p l u n m t p nt t n p . tt t n . m s.

. . p k t n . ll . lg t m n m nt l t g n l g p ng n t m n n . s 3( ) - 5 .

. . m . n m ng g p n t g t t m n mum num n . 6(3) 7.

. . n n . g n. t l n g p n t m ng . 355 37 5.

3. . n. m t pl n g p t ngl . n s pp. 6 6.

. . . . - n t - m n n l t g n l g ng m mum g g p . / 3 l mput n n t ng n - ng n n t .

5. . . . n lg t m t - m n n l t g n l g p ng. n . t . 6^{th} p ng - l g l. 5 7 pp. 33 -3 6 .

6. . . . - m n n l t g n l p ng. . . t l mput n n t ng n ng n n t .

r s i l* orst i l i . oo**

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t i p p r tud t r di io ort o o o dr i o r p it out oop . pro id o r ou d or t r rio) dr i r rti ou d d p t r tio 2) dr i r t ur o rti i proportio to t ir d r d 3) dr i it out u r tri tio . i o tru tio t t t t o r ou d i rio it i ord r o itud .

t is p p r o si r t r i sio l ort o o l r i s o rt r p ( ) it i r (llo to p r ll l s b t o s l loops). ( ) o r pr s ts rti s b p ir is o i t rs ti bo s i t t r i sio l ri . is r pr s t b s q o o ti o s s ts o ri li s possibl b t t ri poi ts b t (poi ts tr l i p rti l r ir tio) o t bo s o . t o ro t s r isjoi t pt possibl t poi ts. r p t or ti t r s ' rt ' ill lso r r to t ir r pr s t tio i ort o o l r i . b r o ports o bo ill b ll its f t b r o ri poi ts i bo is its . o bo is its l r st si l t i i b its s ll st si l t .

---

* r upport d .

** upport d t u tr i r ou i r r t 99 2 d p rti o p t d i tud t i t oo o o put r i d o t r i ri t o i r it u d r t up r i io o r r rr.

s
○

ort o o l r i it p rti l r s p o bo r pr s ti r rt . . poi t li or b is ll ort o o l r i or p rti l r s p . rt o o l poi t r i s b st i i 7 9 6]. o r ort o o l poi t r i s o l ist or r p s it i r t ost si . r o i t is r stri tio s oti t r t i t r st i ort o o l bo r i s 3 6 9 2].

s ll st bo losi ort o o l r i is ll t . bo i bo ol t i b r o b s p r r t ost o o l propos s r s or t r i i t st ti q lit o ort o o l r i . or bo r i s t si s p o rt it r sp t to its r r lso o si r i port t s r o st ti q lit .

ort o o l r i is s i to b i or so o st t t s r o is t ost ( ) or ll rti s ; it is i t s r o is t ost ( ) + (( )) or ll .

t ( ) ot t i i t k o r ll ort o o l r i s o r p t t sp t r tios t ost r stri tl r r stri t . t ( ) b t i t k o r ll r p s it rti s s o ( ). s ( ) s rib s ol bo it i i r ll r p s it rti s s s t t rt s sp t r tio t ost s r t ost ( ).

rst lo r bo s or 3 ort o o l bo r i s r to i r 2]. s o t t i t bo ot tio ( ) $\Omega($ (lo) $^{/}$) t is p p r s o t t

- ( ) $\Omega($ ¯)
- ( ) $\Omega($ / ¯)
- ( ) $\Omega($ /).

t s i pro t r s lts o 2] i t r s irstl r o t lo tor to st blis ( ) $\Omega($ /) s t lo r bo . o l pro t is r s lt i o l o o i bo sp t r tios b i stri tl r r stri t ol s. i ll lso st t s it r o t s t o o itio s ol st blis k r b t opti l lo r bo . r rst r s lt i l s t lo r bo o $\Omega($ /) or ort o o l r i s o $_n$ st blis b i l]. t t proo o o r lo r bo s r b s o t iq s lop i t is p p r t o r li t to r p s it rbitr r b r o s i l o si r tio s o bo sp t r tios r r stri tio .

tr o b t t i b r o b s p r ro t t bo i bo ol is pp r t i l orit s or ort o o l r p r i ; s bl or o r i o t k o r s lts.

t is p p r pr s t t o opti l l orit s or ort o o l r p r i . rst l orit pro s r r stri t ort o o l b r i s it (/) bo i bo ol t ost si b s p r . t iq s is r li tio o t COMPACT l orit o s]

tr d o t ou di o ou d t i u u r o d i ort o o r p dr i . o r ou d r pro d i or 2.

o r ou d	o u	d	od	r p	r r
Ω)	$\mathcal{O}$))	2	r po itio	i p	3 9]
Ω)	$\mathcal{O}$ $\sqrt{}$)	2	i ti d	i p	3]
Ω)	$\mathcal{O}$)		p out	u ti r p	.
Ω)	$\mathcal{O}$))		p out	i p	2]
Ω)	$\mathcal{O}$)		p out	u ti r p	. 3
Ω)	$\mathcal{O}$)	2	r po itio	i p	3 9]
Ω)	$\mathcal{O}$)	2	i ti d	i p	3]
Ω)	$\mathcal{O}$)		p out	u ti r p	.
Ω)	$\mathcal{O}$)		p out	u ti r p	. 3
Ω $\sqrt{}$)	$\mathcal{O}$)		i ti d	i p	]
Ω $\sqrt{}$)	$\mathcal{O}$)	3	i ti d	i p	]
Ω $\sqrt{}$)	$\mathcal{O}$)	3	p out	u ti r p	.
Ω $\sqrt{}$)	$\mathcal{O}$ $\sqrt{}$)		p out	i p	.

or ort o o l poi t r i is i pro t o t l orit s o i r 2] oo 2] o obt i pp r bo s o (( ) /) ( ) r sp ti l . r s o l orit pro s ort o o l r t l r i s it ( ) bo i bo ol t ost o r b s p r ; t r i s r ot ss ril r r stri t or o t rti s bo sp t r tios. ot pp r bo s r t r or it i or r o it o t lo r bo . lso pr s t r ts o bot o r l orit s it o l ss b p r t t ost o i r s i t ol .

t is s tio pro lo r bo s si r p st t $\Omega($ $)$ s i t it t l st 6 rti s o si o t t.

f o o (−) 2 $_{p,q}$ f

- $_{p,q}$ f +
- f f $_{p,q}$ (−) 2 (−)
- f ($_{p,q}$) 36 9

i r t 2] did ot ou t t u r o d p r d d du t ou d o ro t ir o tru tio .

f (k t) t $_{p,q}$ b t j r p ^{p,q} i]; t rst t o prop rti s o t r p r s o i t is p p r. t s lso s o t t 2 – r ot s t s o l r st i l o $_{p,q}$. ss r isjoi t rt s ts it 36. k o ro 2] t t t b r o – s b t is t l st $\frac{d\ S\ T}{n}$ – . si 6 o s o t t 2 t b r o s b t is t l st (– 2) –(– 2)

r lo r bo proo is b s o t t iq lop i] i isti is s t r s s it r rti s r i t rs t b o ri li or rti s r i t rs t b o pl or it r o t s is t s .

( )
f 36

- ( ) – / /
- ( ) – / ( ) / –
- ( ) – ( ) /

f o si r r i o i ri o i sio s . or s o proo ss t t is i isibl b 6; s 6] or proo it o t t is ss ptio .

ss t t (i. . ri li p r ll l to t is) i t rs ts t l st – rti s. t b s t t t ( ) pl i t rs ts o o t s rti s s p r t s t i to t o ro ps o t l st – rti s . ss ptio t l st s o t t s t o ro ps. s s ross t ( ) pl i t s st o t i t l st poi ts it i t r oor i t s. . i t li i t rs ts t l st – rti s – so – i pro s ll l i s b .

ss t t or (i. . pl p rp i l r to t is is or is r sp ti l) i t rs ts t ost – rti s. rt is f o ( ) pl i ll t poi ts i its ri bo oor i t s l ss t . otio o o ( ) pl is lo o s. t b t l r st i t r l l s t t r t – rti s r l t o t ( ) pl . i t ( ) pl i t rs ts t ost – rti s t r r t l st – rti s ri t o t ( ) pl . ll t s rti s lso li to t ri t o (+–) pl . itio o t l st – rti s li l t o t (+) pl . ll t s rti s lso li to t l t o (+–) pl .
ss ptio t r r t l st s b t t rti s o t l t t rti s o t ri t o t (+–) pl so . i t s r t ol s or t ot r t o ir tio s ( ) / i pro s ll l i s.

ss o t t li li or
li i t rs ts t ost – rti s b t t r ists s ( ) pl t t
i t rs ts t l st – rti s. s ( ) pl is s pt ro s ll r to
l r r l s o t li t r i b t i t rs tio o t is ( )
pl it t ( ) pl s ps t ( ) pl . t ti t is
li i t rs ts t ost – rti s b ss ptio . i il rl s bo o
t r or ( ) pl t t splits t rti s i t rs t b t
( ) pl i to t o s ts _ to t l t ri t o it i
o t i t l st – rti s. ss ptio t r r t l st s b t
_ so . ppl tl t s r t i t ir tio
to obt i . t t t r lo r bo s s ollo s

- ( ) pl i t rs ts t l st – rti s so – . is i pli s
  – ( ) – / i pro s
  t rst lo r bo .
- ss t t r rt s sp t r tio t ost ( ). p rti l r
  t r or ( ) ( ) ( ) ( ) or r rt r pr s t
  b ( ) ( ) ( ) bo . i t s r o is t l st ( ) t is
  i pli s ( ) ( ) ( ) 6 6 . i t ( ) pl i t rs ts
  t l st – rti s t s i t rs tio s r isjoi t t r st b t
  l st – 6 i t r l poi ts i t ( ) pl . o –
  – ( ) – ( ) / – i
  pro s t s o lo r bo .
- ss t t t s r o r rt is t ost ( ) ( )
  i i pli s ( ) . _ – + .
  s t t poi t is i its oor i t s tis s _
  ot r is . ot t t ll rti s i _ ross t ( )
  pl t ross it r t (_) pl or t ( )
  pl ll ri poi ts o ll rti s i _ r i si .
  ll t t t ( ) pl s p r t s t rti s i _
  is ross b t l st s. r r o l 2 i t r l
  poi ts t t r i si o t ( ) pl so t l st – 2
  s ross t ( ) pl t o tsi poi t.
  o t s – 2 s st rts t rt i _ t i si poi t
  ross s t ( ) pl t o tsi poi t s t rt i
  t i si poi t. s ross s t (_) pl or t
  ( ) pl t l st t i . s t o pl s to t r t r or st
  t l st 2(– 2) ri poi ts t r or – 2.
  ppl i t t s r t i t ir tio obt i
  – 2.
  o i or t 2 t r or
  – ( ) – / ( ) / .
  t ( )
  ( ) ( ) ( ) / b . it r
  t t ir l i is pro .

o t t r s lt to rbitr r l s o s lo s bot r bi o .

• •• • • •• 2 f

$\overline{k}$] o

f (k t) is ollo s ro o s t or b l ll o ssi t t st blis s t t t b r o pri s it o is proportio l to lo (s . .] or proo).

• •• • • •• $\binom{n}{}$

- fi f
-
- 2 6
- f

2 f

f (k t) ss 32 2 -(+) — r is t lo r bo o i 2. proo splits i to t o s s p i o t r is bi or ot.

32 t l t b t o pl t r p o + rti s. o obt i − itio l rti s $-\binom{n'}{}$ rbitr r s s t t t r s lti r p is si pl . si t ss ptio s o o ri ll o itio s.

32 t l t - + $\overline{\frac{n}{} + -}$ pri it

o s t t $\overline{k}$. t 2 (−) − pri it o s t t $\overline{k}$. si t o itio s o o ri s so t is is possibl . lso (−) 2. t b t r p $_{p,q}$ s i s ppos it s rti s s. r t b i − rti s − s b t t ; b 32 o s o − $\binom{n-n'}{}$ t is b o s t t is si pl t r s i r . si t ss ptio s o o ri ll o itio s.

• •••••• •• ( ) $\Omega($ $\overline{\ })$ ( ) $\Omega($ / $\overline{\ })$ ( ) $\Omega($ / $)$

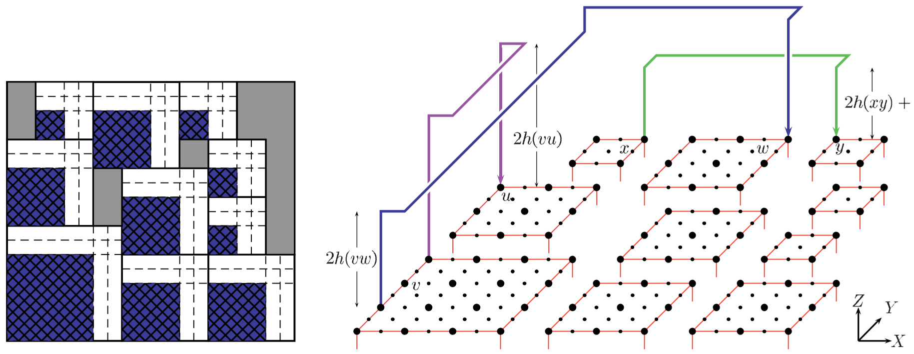
f (k t) t b t r p o 3. r i o o t i s r i o t s s ol − / ( ) / b or . t r i o s sp t r tios bo b t so o s t r i o i t s s ol − / ( ) / ‾. t r i o is stri tl r r stri t t r rt i s s r . s t r i o is stri tl ( ) r r stri t s ol t − ( ) / . si t bo s o k o ro 3 o obt i s t r s lts si is o st t.

t ollo i i t o o str tio s. rst o r t s b r i s it s ptoti ll opti l ol t s o o r t s r i s it o t r stri tio s o rt bo s t t s ptoti ll opti l ol .

t ollo i l orit or pro i ort o o l r i s rti s r i iti ll r pr s t b sq r s i t ( ) pl s r ro t bo t pl . rti s r t t i t i sio to or b s. or sp r so s i r o l si pli o str tio it l r r o st ts.

. pr s t rt b sq r $_v$ o si l t 2 $\overline{\quad}$ ( ) + 2.
2. ositio t sq r s $_v$ i t ( ) pl it t sq r p ki l orit o l it ri r 3].
3. or rt r o t top t o ro s ro $_v$ t t o ri t ost ol s ro $_v$. rti s r o isjoi t; s i .).
. ir t s rbitr ril ssi iq ports (ports tr i t ir tio) t bot it oor i t oor i t .
. o str t r p it ( ) . or ori t s t to ( ) i t ports o t o t r i t s ol or i t ports o t o t r i t s ro .
6. rt olo r t r p it ( ) + olo rs or rt ( ) olo r orr spo i to s t h( ) .
7. or ori t o str t ro t or s ollo s. ppos t ports o ssi to oor i t s ($_X$ $_Y$) ($_X$ $_Y$) r sp ti l . o t t it o o t ollo i o r or si b ro t s s ill str t i i . .
   - $_X$ $_X$
     ($_X$ $_Y$) ($_X$ $_Y$ $2h$( )) ($_X$ + $_Y$ $2h$( ))
     ($_X$ + $_Y$ $2h$( )) ($_X$ $_Y$ $2h$( )) ($_X$ $_Y$)
   - $_Y$ $_Y$
     ($_X$ $_Y$) ($_X$ $_Y$ $2h$( ) +) ($_X$ $_Y$ + $2h$( ) +)
     ($_X$ $_Y$ + $2h$( ) +) ($_X$ $_Y$ $2h$( ) +) ($_X$ $_Y$)
   - $_X$ $_X$ $_Y$ $_Y$
     ($_X$ $_Y$) ($_X$ $_Y$ $2h$( )) ($_X$ + $_Y$ $2h$( ))
     ($_X$ + $_Y$ + $2h$( )) ($_X$ + $_Y$ + $2h$( ) +)
     ($_X$ $_Y$ + $2h$( ) +) ($_X$ $_Y$ $2h$( ) +) ($_X$ $_Y$)
. l r sq r s r pr s ti rti s i to b s b t i t ir si p r ll l to t is.

orit i r t 2] i i i r i pirit ut u qu r o i $\mathcal{O}\sqrt{\quad}$) or rt r u ti i $\mathcal{O}$) o u dr i .



(a) (b)

• •• • • •) qu r p i) outi d .

• • • • • • • • •

f (/) (/)
$\Omega($ $)$

f (k t) sq r s $_v$ r $_v(2$ ( ) + 2)
+ ( ). l orit o 3] p ks sq r s it tot l r o i
2 3 2 r t l . o t sq r s $_v$ ( ) b p k i r t l
it si (2 3 + ( )) (2 + ( )) ( ) ( ).
t i r o is ( ) t i t o t r i bo t
( ) pl is lso ( ). p i rti s s i t 2 .
t tot l i t is ( ) i pro s t bo o t ol .
s r o rt is 6(2 ( )) 2 ( ) + (( ))
o str tio t r r t ost si b s p r ro t .
ti o s i st o t l orit is t rt olo ri o
i t k s (( )) (( ) ( )) ( ) (/) ti .

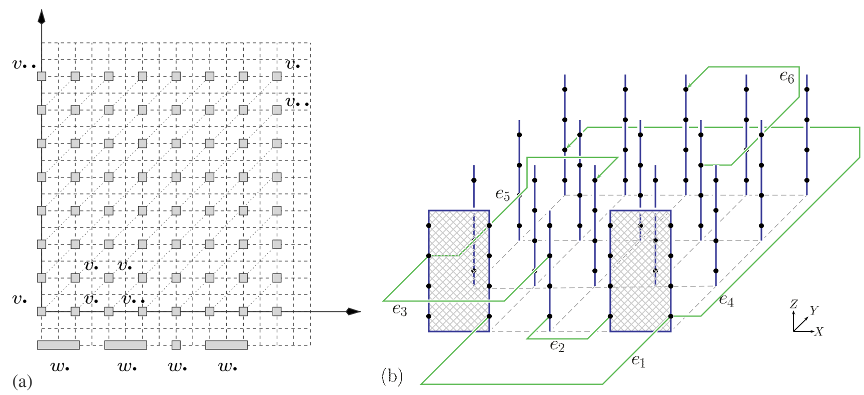
ro ti t s bo b lo t rti s 2 r r stri t r
ti s r possibl ; s 6] or t ils. r r tl i l] lop
t iq to i pl t st ps 6 o o r l orit or i tl . it
t is t iq t ti o pl it o o r l orit r s to (lo)
t ol r s s to 3 / + (/).
r o t i l s t ro 6 b ro t ro t
it iq i t t t o r ll i t is ( ) obt i t
ollo i r s lt.

• • • • • • • • •

( ) ( )
ﬁ

o s o o to r t ort o o l r i s o si pl r p s t t o l ( ); t rti s bo sp t r tios r ot ss ril r r stri t ; rti s r i iti ll r pr s t b poi ts or li s ts i t ( ) pl . rti s r t t i t i sio to or li s or r t l s. t .

. $_{ig}$ ( ) $_{mall}$ − $_{ig}$. $_{ig}$ to b t s t o s it bot poi ts i $_{ig}$ $_{ut}$ to b t s t o s it tl o poi t i $_{ig}$ $_{mall}$ to b t r i i s.
2. or rt $_{mall}$ r pr s t b 2 2 r t l $_{v}$.
3. or rt $_{ig}$ ( ) ( ) r pr s t b 2 ( ) 2 r t l $_{v}$.
. t $_{l}$ b t rti s i $_{ig}$. or positio $_{w}$. i t ( ) pl s t t its top l t or r is t ($^{j-}_{i}$ 2 ($_{i}$) 2). o ll t s r t l s tt to t is b t ot r.
. tr rti s o r s t t $_{mall}$ s tl rti s. (is st p is o l o to s t s riptio .)
6. ort t rti s i $_{mall}$ b o i r si r s N − . or − i + t pl t botto l t or r o $_{v}$. t (2(+) o 2 2).
7. or rt r o t top t o ro s ro $_{v}$ t t o ri t ost ol s ro $_{v}$. rti s r o isjoi t; s i . 2.
. ri t i $_{ut}$ ro t poi t i $_{ig}$ to t ot r poi t. ri t ll s i $_{mall}$ rbitr ril .
9. or ori t $_{ut}$ ssi port t $_{v}$ it oor i t s to i s t t t ost s r ssi to port o rt i $_{ig}$.
or $_{ut}$ $_{mall}$ it poi t $_{mall}$ s s t iq port it oor i t s t $_{v}$.
. o str t r p it ( ) $_{ut}$ $_{mall}$. or ori t s ( ) t to ( ) i t ports o t o t r i t s ol or i t ports o o t r i t s ro .
. rt olo r t r p it ( ) + olo rs or rt ( ) olo r orr spo i to s t h( ) .
2. or ori t $_{ut}$ $_{mall}$ o str t ro t or tl s s rib i t l orit i t. 3. .
3. o str t ro t s or s i $_{ig}$ b op i t 2 b l o t o t o pl t r p lop i]. or pr is l split $_{ig}$ i to t ost $_{ig}$ − t i s s t t o t o s it i o t i r or to ross ssi o pl or t i ro t i t t i it t ost 2 b s i t is pl .
. l r poi ts/li s r pr s ti rti s i to li s/r t l s b t i t ir si s p r ll l to t is ro t i i to t i oor i t s i t r i . lip ll s ts o s t t o rl p t bo o poi t.



• •• • • •) rt out . . . . o to $\bullet_{ig}$.) d rout . . d . o to $\bullet_{ig}$. d . o to $\bullet_{ut}$ d . o to $\bullet_{mall}$.

• • • • • • • f 6 -2

f (k t) o str tio t r r t ost s i $\bullet_{ut}$ ssi to t s port i t ro it oor i t -2. t b ol it oor i t t r t ki t poi ts it ti oor i t s. t b ro t t s o ti oor i t . r s lt ol s i prop r ro / ol s t ost 6 s t t s port i t t ro / ol . is b s o b l i t rti s i $\bullet_{mall}$ b ssi to poi ts i prop r ro s/ ol s si t t t t t i r o rt i $\bullet_{mall}$ is t ost .

• • • • • • • • • f ($\overline{}$) ($\overline{}$) f

f (k t) st rt it 2 pl s 2 + 2 pl s. s i $\bullet_{ig}$ p to $\bullet_{ig}$ 2 pl s b t o s o t t $\bullet_{ig}$ 2. r rt olo ri o r q ir s t ost ( )+ olo rs so to ro t t s i $\bullet_{ut}$ $\bullet_{mall}$ t ost 2 6 pl s. o ro t t s i $\bullet_{ig}$ t ost $\bullet_{ig}$ 2 pl s i is s ll r b 2 so t i t o t r i bo t pl is t ost 32 . bo i bo is t r or 2 (- + 2) 32 + ( ) ($\overline{}$). ti o s i st o t l orit is t rt olo ri o i t k s (( )) (( ) ( )) ( ) ($\overline{}$) ti . ri i ll t r r t ost si b s p r ro t t ost o r b s p r ro t or s i $\bullet_{ig}$. ri t lippi st p t rst l st s t o is lipp t s r s t ost o r b s.

t iq o i l] orks or st ps o t is l o rit s ll r i t ti o pl it o o r l orit to (lo) r si t ol to 9 ¯+ (¯).

r s t b r o b s to t r b s i llo i r s i ol . t t o str tio r tl si pli s positio t rti s s li s i ri ssi to iq i t ro t s b or b t o itti t i l s t. i t is t t i t pt r bot 2 .

• • • • • • • • •

( ) ( )

is l orit is p rti l rl propri t or ltil r s t r r o rti l s ts (' ross ts); s].

• •

t is p p r pro i t i pp r lo r bo s or t r i sio l ort o o l bo r i s.

p rti l r s o t t l orit to r t t r i sio l ort o o l r i s t t bo sp t r tios or r r r stri t ot o b tt r t bo i bo ol o $\Omega($ / $)$. t l orit t t t s t is bo i. . o str ts t r i sio l r r stri t ort o o l b r i s it bo i bo ol (/).

t r r o r stri tio s o t r i t s o t t o l orit o b tt r t bo i bo ol o $\Omega($ ¯$)$. s o l orit t t t s t is bo i. . o str ts t r i sio l ort o o l r i s it bo i bo ol (¯).

ollo i op probl s r i to b st i

- bl s sts tr o b t t b r o b s p r t bo i bo ol . s tr o b pro ? t r lo r bo s or r i s r s r llo to o l o or t o b s?
-] i l o str t r i s o $_n$ it t ost o b p r s t t rt bo s s r 2 + . or si pl r p t is i l s r i it t ost o b p r b t t is r i is r r stri t o l i t i i r is ( ). o s r r p t r i sio l ort o o l r i t t is bot r r stri t s t ost o b p r ?
- t pp r bo s o t i b r o b s p r i ort o o l r i s it opti l ol b i pro ? p rti l r ( ) o s r r p b r r stri t b r i it (/) o l (b) o s r r p 3 b r i it (¯) o l ? ot t t $_n$ (/) (¯) ol 3 b r i].

• ••••••• •••

. . r . d . or. u ti r rid ddi or . *g* *t*) 29 99 .
2. . o d . p r. *st* *t* . o i o 992.
3. . i d . r ppro to 3 ort o o o dr i . it id] p 3 3.
. . i d d . . ro o ori i pro i t t iqu o o oro d r di . i port 2 3. p rt t o o put r i i r it o t r oo d 2 .
. . i d . r r . it id d . i t . ou d or ort o o 3 r p dr i . *g* 3) 3 9 999.
. . i d . i d . . ood. r i io rt o o r p r i it pti o u . i port 2 2. p rt t o o put r i i r it o t r oo d 2 .
. . o o . rt or . o d . . i t . u d i 3 di io ort o o r p dr i . r to i] p 9 .
. . i tti t . tri i d . r iu. p it pu ppro to 3 ort o o dr i . it id] p .
9. . d . tir d . it id . t iqu o o o oro d rd i or t r di io ort o o r p dr i . *t* *ss* *g* *tt* *s* 2) 9 3 99 .
. . d . o i d . it id . r di io ort o o r p dr i orit . *s* *t* *t* 3 3) 2 .
. . . o . t r proo o or u o d ou i u i). *t* *s* *t* *st s*) 23 3 9 .
2. . i r . o ur d . u u i. r p ddi o t r di io od . *st* *s* *t* *t* *s*) 9 3.
3. . it d . ri r. opti ou d or t o di io i p i . *t* *s* *t* *s* *t* p 3 . 9 .
. . r to i ditor. *s* *g* o u 3 o *t* *t s* *t* . pri r r 999.
. . u ot . i ip d . r . uj r p . *t* 2 2 9 .
. . p o t d . o i . r t ort o o r p dr i i t r di io . *g* 3) 999.
. . it id ditor. *s* *g* o u o *t* *t s* *t* . pri r r 99 .
. . . ood. orit or t r di io ort o o r p dr i . it id] p 332 3 .
9. . . ood. u ti di io ort o o r p dr i it o . r to i] p 3 322.
2 . . . ood. orit d op pro i t r di io ort o o r p dr i . . d . i p o ditor *st* *s* *s* *t* *g* *t* *s* p . urti i r it o o o rt 999.

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•• • • • • • • • • • • • • • •

c o o o

m r t r 3 p

nakano@cs.gunma-u.ac.jp

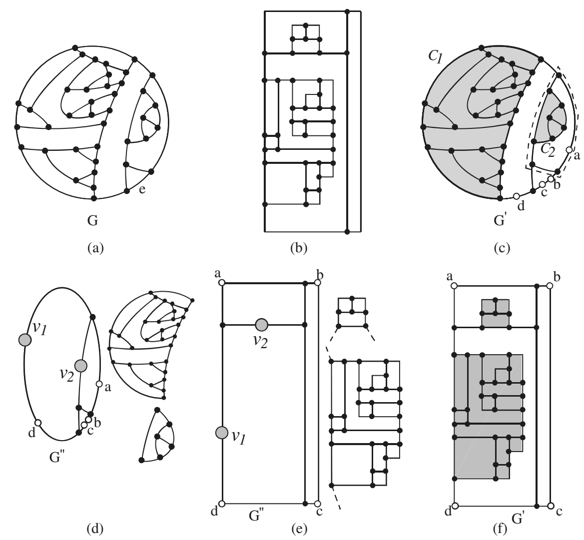
• • •••••• ort o o l r o pl r p • r o • t t pl r m rt m pp to po t r q o lt r t or o t l rt l l m t t o o ot ro pt t t r ommo . r t t o l pl r r p t t m m m r o r or l ort o o l r . t k o l or t m to ort o o l r r t m • (• ˙ / ˙ $\sqrt{\text{lo} \bullet}$ or pl r p t • rt . t p p r l r t m l or t m to ort o o l r o o t pl r p t t m m m m r o .

• •• • • • • • • • •• •

t o pl p G o G t t pl c c t pp to po t c qu c o lt t o o t l t c l l t t o o ot c o c pt t t co o . t o o l tt ct uc tt t o u to t u ou p ct c l ppl c t o c cu t c t c tc. 3 . p t cul to o t o o l t t u u o .

o pl p G t llo to c oo t pl t o t o o l o G t t u u o co pl t . o p t l o t c o t o o l o pl p G t t u u o $O(n$ lo n $O(n$ / $\overline{\text{lo } n}$ t p ct l u l t llo to c oo t pl n t u o t c G. uc t u o t o o l p o l to u co t flo p o l . t ot l l t l o t o o o t o o l o pl p t p u l ll u o o 3 co ct cu c pl p l t l o t o o o t o o l t t u u o . t t o l pl p t t u ou o l o t o o l .

○



• •• • • • pl r p t ort o o l r .

t p p l t ult l t l o t to o t o o l o co ct cu c pl p t t u u o .

o t o o l c t o c c ct l c ll *t* . pl p G uc t t t t t o o t l t c ct ul o G uc p ct ul . o ou l o t to uc t o t o o l p o l to t ct ul p o l .

outl o ou l o t llu t t . . pl p G o . (t t t uctu o o c cl G t l t t t uctu put ou u t c a, b, c d o t o o t out ou o G l t G t ult p . ou u t c t c cl . (c . t co t ct c o o c cl C , C , t (. (c to l t o . (o t t t ult p G ct ul o . (. l o o t o o l o t o c cl C , C , t cu l (. ((. tc t o t t o t o o l o G o . (. pl c t u t c a, b, c d t o G t ll o t o t o o l o G o . (.

t o t p p o ollo . ct o 2 o t o p t o ult. ct o 3 o t t uctu o o c cl G. ct o p t l o t to o t o o l t t u u o .

t G co ct p t n t c . co ct t c x y ot $(x, y$. o t v t u o o o v G. t o G t t G c ll . t t $\kappa(G$ o p G t u u o t c o o l ult co ct p o l t p K . t t G t $\kappa(G$ k.

p t c t pl o t t o t o t ct o t c ll c pt t t to c t ot c t. p pl p t pl . pl p t pl to co ct o c ll . t t o c cloc c cl o t o t ou o t c . ot t co tou o t out c o p G $C_o(G$.

o pl c cl C pl p G ot $G(C$ t pl u p o G C (clu C . t t c cl C C pl p G t $G(C$ $G(C$ o co o t . cl C C t t C C o co o t .

c c t to ctl o t o pl c cl C loc t out o C c ll o t c cl C t t o C to c t l c t c ll t o C. pl c cl t ctl k l c ll k . o k l c cl C t k u p t o C C t t k l t c c ll t t t o C.

t o pl p G o G t t pl c c t pp to po t c qu c o lt t o o t l t c l l t t o o ot c o c pt t t co o . po t c t ct o c ll . ot $b(G$ t u u o o o t o o l o G. o t o o l o G t ctl $b(G$ t .

t o pl p G o G uc t t c o o t l o t c l l t c c ct l . u ct ul o t o o l c t o c c ct l . o G . (ct ul . o G . (ot ct ul ut o t o o l . ct ul D o G t ou co o t ct l co po to $C_o(G$ t c o t o o $C_o(G$. c ll t ou t c t t o D. ollo ult o ct ul o .

t G t t t t t t t t a, b, c, d t t t $C_o(G$ G t t t t a, b, c, d G t t t

t t t t t t

3 3 t t t t

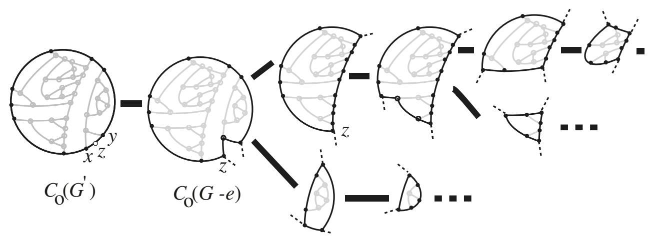
t t t G t t t
G t t G
t

• • • • • • •• • •• • • • • •

t G co ct cu c pl p . o p o t ct c cl C_a C_d
G C_d c ll t o C_a (C_d t 2 o 3 l c cl
($G(C_d$ p op u p o $G(C_a$. ot t t c G co ct
t t o l c cl c pt t o l l c cl $C_o(G$. o
c oo e $(x, y$ o $C_o(G$ pl c e t t o $(x, z$
$(z, y$. t G' t ult pl p . (ot t t o $G - e$ t t pl
u p o G o t o G l t e $C_o(G - e$ 2 l c cl o
G' o $C_o(G - e$ ot 2 l c cl o G. t $D_e(C_o$ C C
c t c cl o $C_o(G'$ ot co t z . c cl C_c $D_e(C_o$ c ll
o $C_o(G'$ (t p ct to e C_c ot loc t o
ot c cl $D_e(C_o$. c G co ct cu c pl p $C_o(G'$
ctl o c l c cl $C_o(G - e$ (t p ct to e . (2.
cu l o c c l c cl C_c t c l c cl ollo .
t ollo t o c .

C_c 2 l c cl .
oo l t o C_c z. t $D_z(C_c$ C C c t c cl o
C_c ot co t z . c cl C_{cc} $D_z(C_c$ c ll o C_c (t
p ct to z C_{cc} ot loc t o ot c cl $D_z(C_c$. c
G co ct cu c pl p C_c t o t o 3 l c l c cl .
$(C_c$ o 3 l c l c cl $G(C$ c F co t t t o
l t c C_c ctl o 3 l c l c cl ot .
t C_c 3 l c cl .
t $D(C_c$ t t o ll c t c cl o C_c. c cl C_{cc} $D(C_c$
c ll o C_c C_{cc} ot loc t o ot c cl $D(C_c$.

ot c o ll c l c cl o C_c p t c ot .
t t o o c c l c cl o c c l c cl cu l
tu ll t (c c l t t uctu o c cl G p t
" lo c l t " T_g o 2. c u o t c o c o e
z T_g o t o . c oo t (ut o T_g.
t o l to o l t o
c uc t t uctu T_g o c cl t t co tou o c
c co t t u o t .
o o t ollo . o t o o l o G c cl
C G t l t ou co co . . pol o l t c o l
. c G cu c uc co u t t ot l t o C.
u t ollo ct o o t o o l o G.



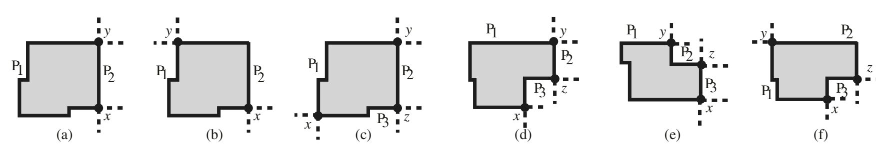
• ••••• l • ′ lo l tr • $_g$.

t l t ou u t pp o $C_o(G$.
t l t t o u t pp o c 2 l c cl G.
t l t o u t pp o c 3 l c cl G.

• • ••••••••• • ••• ••• •

t ct o l t l o t to opt l o t o o l o co ct cu c pl p . u t t lo c l t T_g o co ct cu c pl p G. o t o .
t C 2 l c cl t t t o l t c x y P P t cloc co tou p t o x to y o y to x p ct l . opt l o t o o l D o $G(C$ o $(P$, P o o t ollo ou op lfl t ct D. (. 3(. tu t l D t o co o P .

t t c l op lfl t t upp t x.
t o o t l op lfl t t l t t x.
t t c l op lfl t t lo t y.
t o o t l op lfl t t l t t y.

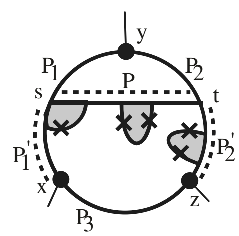


• ••••• ll tr t o or l r .

l o opt l o t o o l D o $G(C$ o $(P$, P o o t ou op lfl p ct l . 3(t ct D.
t C 3 l c cl t t t l t c x, y z pp cloc t o P P P t cloc co tou p t o x to y o y to z o z to x p ct l . opt l o t o o l D o $G(C$ o $(P$ o o t op lfl p ct l . 3(c t ct D. l l o t o o l o $(P$, P , $-P$ $(P$, P , $-P$ $(P$, P , $-P$.(. 3((.

o o c c cl C $C_o(G$ co po to t T_g t t $G(C$ c t p o l otto up co put t o o T_g. o t otto up co put t o l o co put t S_C o t o t c cl $G(C$ co t o ℓ 2 l c cl ℓ 3 l c cl o o ℓ ℓ . u $b(G(C$ 2 ℓ + ℓ ct 3.2 3.3. t o t t $G(C$ l t l t o l u 2 ℓ + ℓ . u $b(G(C$ 2 ℓ + ℓ ol .

t otto up co put t o cl c co tou p t o c c cl t t. tu t l k co p t c c to k co . P P t t o co p t ollo . t



• ••••• ll tr t o or •.•. tr .

x, y, z t t l t c o 3 l c cl C P P t cloc co tou p t o x to y y to z p ct l . u t t s t t c o P P p ct l l t P' t u p t o P o x to s P' t u p t o P o t to z. (t p t P o s to t uc t t t l t o P c o $G(C$ ($G(C$ o c l c cl o 2 co p t o P, P' o P' t t p t co t o P', P, P' c ll P P t . pl llu t t . . tut l o l t o c c to tu t t s t o P P t o x to z.

t otto up co put t o o t t t ollo co t o (c (c ol .

(c c cl C t l t o o 2 co p t .
(c2 o c cl S_C co t o co p t o C.
(c3 o 2 l c cl C C co p t P t $G(C$ t S'_C o t o t c cl co t o o P co t o ℓ' 2 l c cl ℓ' 3 l c cl uc t t 2 ℓ' + ℓ' $b(G(C$ − .
(c o 2 l c cl C C co p t P t t ot co tou p t P 2 co p t $G(C$ o t o o l l o $(P$, P .
(c o 3 l c cl C C co p t P t $G(C$ t S'_C o t o t c cl co t o o P co t o ℓ' 2 l c cl ℓ' 3 l c cl uc t t 2 ℓ' + ℓ' $b(G(C$ − .
(c o 3 l c cl C C o 2 co p t P t $G(C$ o t o o l l o $(P$.
(c o 3 l c cl C C 2 co p t P o P P t t $G(C$ o t o o l l o $(P$, P , $-P$

(c o 3 l c cl C C 2 co p t P o P P t
t $G(C$ o t o o l l o $(P, P, -P$
(c o 3 l c cl C C co p t P P o P P
t t $G(C$ o t o o l l o $(P, P, -P$.

o pl t otto up co put t o t ollo ou c .
C 2 l c cl o c l c cl .
t x, y t t o l t c o C l t P P t cloc co tou
p t o x to y o y to x p ct l . o $G(C$ C c o
2 l c cl C $G(C$ p op o C t C l
c l c cl .
• C t S_C C . ct 3.2 o t o o l o
$G(C$ t l t t o .
t o uc t o o P c l co t uct
o t o o l o $G(C$ l o $(P, P$. l l c co t uct
o t o o l o $G(C$ l o $(P, P$ $(P, P$ p ct l . u
$G(C$ c t p o l o t o o l .
t c co tou p t o C
cl 2 co p t . o t o (c (c ol c co tou
p t o C 2 co (c (c ol c C ot 3 l c cl .
C 3 l c cl o c l c cl .
t x, y, z t t l t c o C l t P P P t cloc
co tou p t o x to y o y to z o z to x p ct l . o
o ll o C o $G(C$ t t $G(C$ C o t
uc co ct p co t t l t o t o c P, P, P c
ot C c l c cl co t ct o .
• C t S_C C . ct 3.3 o t o o l o
$G(C$ t l t o .
o t uct p G' o $G(C$ o
u t c v o P . o t ult p G' o c cl (c G
o c l c cl t p ct to co t c x, v, y, z t G'
ct ul t t co t c $x, v, y, z.$ ct ul
l o o t o o l o $G(C$ l o $(P$ u ctl o
(co po to v . l l c l co t uct o t o o l o
$G(C$ l o $(P$ $(P$.
o $G(C$ o o t o o l l o $(P, P, -P$ c t
t l t t o o l o P . l l $G(C$ o o t o o l l
o $(P_i, P_j, -P_k$ o i, j, k , 2, 3 .
t c co tou p t o
C cl co p t . o t o (c (c2 ol c co tou
p t o C co (c3 (c ol c C ot 2 l c cl (c ol
c oo S'_C ϕ (c ol c $G(C$ o t o o l l
o $(P$ $(P$ $(P$ p ct l t o o (c (c ol c
$G(C$ o 2 co p t .
C 2 l c cl o o o c l c cl .
t x, y t t o l t c o C l t P P t cloc
co tou p t o x to y o y to x p ct l . $G(C$

c co t x y t C o 3 l c l c cl ot C
ctl o 3 l c l c cl c co t ctl o l t c o C.
u C t o t o 3 l c l c cl .

t C , C , , C_ℓ t c l c cl o C. u t t o C_i i l
l S_{C_i} o t $G(C_i$ c t p o l
co t o (c (c ol . t ollo ou u c . oo o
(c (c o tt .

C o c l c cl o 2 co p t o C.

• C o t o (c2 t t o c cl $S_{C.}, S_{C.},$, S_{C_ℓ}
co t o C. l o c G cu c C t o t to c cl
$S_{C.}, S_{C.},$, S_{C_ℓ}. t S_C C $S_{C.}$ $S_{C.}$ S_{C_ℓ}. u to
t o uc t o .

t co t $G(C$ o t o o l
l o $(P , P$. o t uct p o $G(C$ t o
u t c v, w o P ut ot o c l c cl o C. co t ct c
$G(C$, $G(C$, , $G(C_\ell$ to t c v , v , , v_ℓ p ct l . . (
(. o t ult p c cl ct ul D t
t co t c x, v, w, y. . (c . t C 3 l c l c cl
C' t o t o o l o $G(C'$ l o $(P'$ P'
t co tou p t o C' ot o C cu . co t o (c
(c $G(C'$ l uc . t o t o o l o c
2 l c l c cl $G(C_i$ l o $(P_i'', P_i''$ P_i'' t co tou p t
o C_i ot o C cu . co t o (c $G(C$ l uc
. ll p tc t o $G(C$, $G(C$, , $G(C_\ell$ to D.
. (. p tc o 2 3 l c l c cl l o co ctl
o . . . u c co t uct o t o o l
o $G(C$ l o $(P , P$. l l c co t uct o t o o l
l o $(P , P$ $(P , P$ p ct l .

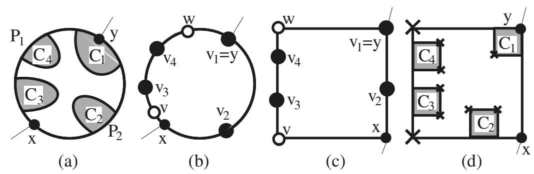
t c co tou p t o C cl 2 co
p t .

C ctl o c l c cl o 2 co p t o C
t c l c cl 2 l c cl .

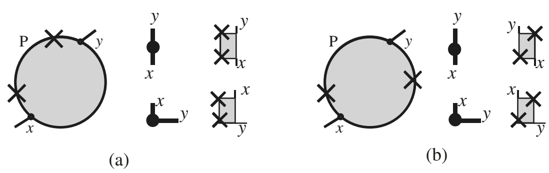
• C t C t 2 l c l c cl co p t
o C. co t o c . C 2 co p t o C t t S_C $S_{C.}$
$S_{C.}$ S_{C_ℓ}. t c o ot to t o uc . C
co p t o C t (c3 $G(C$ t $S'_{C.}$ o t o t c cl
co t o o C co t o ℓ' 2 l c cl ℓ' 3 l
c cl uc t t 2 $\ell' + \ell'$ $b(G(C$ $-$. o t o (c2 t t o c cl
$S_{C.}, S_{C.},$, S_{C_ℓ} co t o C. t S_C C $S'_{C.}$ $S_{C.}$ S_{C_ℓ}.
t c to t o uc o .

tt . l to t p ou c .

C 2 co p t o P t P 2 co p t
P co p t . C 2 co p t o P t P co p t
P 2 co p t . C co p t o P t P 2 co
p t P co p t . (t c c o t o



• •• • • ll tr t o or 3(.



• •• • • ll tr t o or p t .

P o P . C co p t o P t P co p t P 2 co p t .

C ctl o c l c cl o 2 co p t o C t c l c cl 3 l c cl .

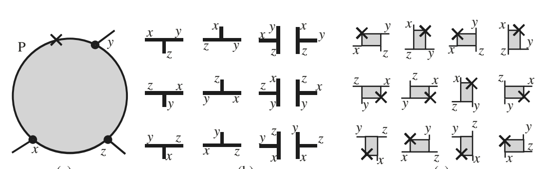
t C t 3 l c l c cl o 2 co p t o C. u t t C y t C l t . t P t co tou p t o C o P P t co tou p t o C o P .

• C co t c .

C P P t t t S_C C_S S_C. S_C. S_{C_ℓ} C_S t 3 l c cl co t o t P P t t o P P ot co t C . t t o o t (c2 C_S t o t to c cl S_C. . t c to t o uc o o C_S. (. ((.

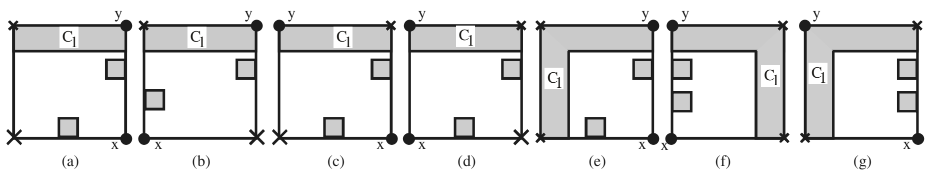
t C o P P t t (P 2 co p t (P 2 co p t o (P co p t P co p t t t S_C S_C. S_C. S_{C_ℓ}. t c o ot to t o uc . (. ((.

t C o P P t t (P co p t P co p t o (P co p t P co p t . (c $G(C$ t S'_C. o t o t c cl co t o o C



• •• • • ll tr t o or p t . (ot t r om tt .

co t o ℓ' 2 l c cl ℓ' 3 l c cl uc t t 2 $\ell' + \ell'$ $b(G(C$ $-$. t S_C C $S'_{C.}$ $S_{C.}$ $S_{C_\ell}.$ u t c to t o uc o . (. ((. tt . l to t p ou c .



• •• • • • ll tr t o or 3(.

t (P 2 co p t C o P P t (P o 2 co p t C P P t o (P co p t P co p t C o P P t t P 2 co p t . (. (((p ct l . t (P co p t P co p t C o P P t (P co p t P o 2 co p t C P P t o (P co p t P co p t C o P P t t P co p t . (. ((c (c p ct l . t P co p t P 2 co p t C o P P t t P co p t . (. (. l P l l .

C t o o o c l c cl o 2 co p t o C. tt

C 3 l c cl o o o c l c cl .

t x, y, z t t l t c o C l t P P P t cloc co tou p t o x to y o y to z o z to x p ct l .

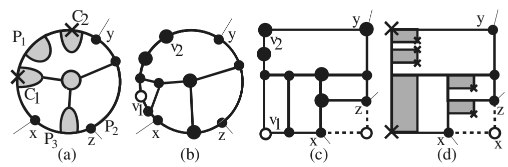
• C C o c l c cl o 2 co p t o C t t S_C C $S_{C.}$ $S_{C.}$ $S_{C_\ell}.$ t c to t o uc o . t t S_C $S_{C.}$ $S_{C.}$ $S_{C_\ell}.$ t c o ot to t o uc .

$G(C$ o c l c cl o 2 co p t o C t $G(C$ o t o o l l o $(P$ $(P$ $(P$ p ct l . (t c to t o uc o .

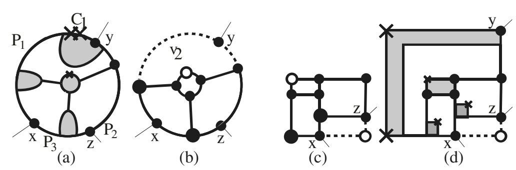
t $G(C$ o t o o l l o $(P$ o l $G(C$ c l c cl o 2 co p t o P . l l c t t $G(C$ o t o o l l o $(P$ $(P$.

C o c l c cl o 2 co p t o C t $G(C$ o o t o o l l o $(P, P, -P$ c o c c to t o o P t o uc o o P .

$G(C$ o t o o l l o $(P, P, -P$ o l (C t o c l c cl o 2 co p t o P o C c l c cl 2 co p t o P (C o P P t . (o t uct o o tt . . .



• •• • • • ll tr t o or .



• •• • • • • ll tr t o or .

C o c l c cl o 2 co p t o C t
P P P co p t . t t (C t o o o
c l c cl o 2 co p t o P o C c l c cl
2 co p t o P t P cl 2 co p t . t C
ctl o c l c cl co p t o P t P cl
co p t . t P cl co p t . cl P
l l .
o ou l o t to opt l o t o o l .
t o l to o t l o t o u
l t .

t o o l (G

oo e o $C_o(G$; lo c l t T_g;
2 o t otto up co put t o ;
3 l c cl o 2 co p t o $C_o(G'$ po
l ;
o t ollo u t l G ctl ou t c o t o.
o c l 2 l c cl C 2 co p t o G pl c
$G(C$ t qu l co t t o t c o t o o G .
o c l 2 l c cl C co p t o G pl c
$G(C$ t t o t o.
o c l 3 l c cl C co p t o G pl c
$G(C$ t qu l co t o t o t o o G .
ut t c o t o o t e.
l c cl C , C , , C_ℓ;
t G'' t p o G' co t ct c $G(C_i$ i
, 2, , ℓ to t v_i;
ct ul $D(G''$ o G'';
o c i , 2, , ℓ l o t o o l $D(G(C_i$ o
$G(C_i$;
tc t $D(G(C_i$ i , 2, , ℓ to $D(G''$ to t o t o o
l o G; (. ((.
.

t t t
t t

3 . tt t . ott . r i i
i i i i
ro . o ork op o l or t m t tr t r pr r
(3 2.
. r . m i i
i i i i ro . o r p r ’
pr r (2 2 .
. r . m i i fl i i i
i i ro . o r p r ’ pr r
(2 22 .
. t i i i i l or t
m (32.
. t . i i
ro . o ’ 3 pr r (3 .
. . m . k o . k i i
ro . o ’ pr r (2 .
l o omp t t o l om tr or ppl t o (2 3
22 .
. . m . k o . k i i
i i i i o r l o
r p l or t m ppl t o 3 (3 2.
. . m . k o . k i
i i ro . o ’
pr r (2 .
. m i i i i i i
. omp t. (2 .
. om i (. . . o
. . . rt ro r r p or m r (
3 .

i . . to io o i i . oo

r p rt t o o put r
r t o
2 u tr
{ben,symvonis,davidw}@cs.usyd.edu.au

t p p r tro u u r o t qu or t r t o t r - o ort o o r o u r r p . p t r t ort or t r - o ort o o r p r u u r o ur t to pro t r p r or . p r or o t r t o t pro u r t u t t p r t tu . ur t t t r t r o t ou o o u t r u u r o p r t r u t . t t o r p u tt- t t 3 our r t or t pro t o t t r t r % 3 % % % 9% r p t .

o i t o i - p it i t r oor i - t to t r it t i -p r ll l t r i t poi t . o r p pl t rti t ri -poi t ro t t lo q o o ti o t o ri -li . r llo to o t i o l i t r t t o o rt .

or r it - ort o o l r i o r p G ot $D(G)$ i . r i it o or t b p r i ll b . r p -t or ti t r ' rt ' l o r r to t ir r pr t tio i r i . t rt v t i ir tio t i i t it v r ll . l rl ort o o l r i o l i t or r p it i r i . - ort o o l r p r i t i i 2 7 9]. r pr ti rt ri - o - ort o o l r i o r itr r r r p l o o i r ( ]).

o i o o i r i i t i i i -p r ll l o i lo t r i . ollo i t ti rit ri r t o t o o l propo r or t q lit o i r i .

i i i t o i o ol .
i i i t i or r r o p r .
i i i t i or r l t o .

i tr i t or r t io o t orr po i 2- - r r - lt opti i i o t rit ri i - r]. t i ti l orit

○

or - ort o o l r p r i t r i pp r t tr o t t t ti rit ri i p rti l r t t o i o ol t - i r o p r ( ]).

pit t t t t t r i pro t i ti l orit po - r l ir l t or ti l prop rti t l r l il to o i t to t r t ti prop rti o t r p i i li . poor i- l q lit o r i pro rr t l orit ttri t to t r p -t or ti t o i t plo . t ir ort to r t i t r tio - r r i or or t- i p t r p t pro or t- r i t r p r i tt r .

o t-pro i r t t iq lp r ti t i it tio . r i pli t r i il i t i i ir t or ti prop rti t i r o p r ro t t o i o ol . t i p p r i tro r o t iq or t r t o - ort o o l r p r i . p r or o t r t o r i pro r l i ti l orit i t l t i t i p ri t l t . t t iq or 2- ort o o l r i lop öß i r 6] i].

t ollo i itio . i l t o X, Y, Z . p o ir tio i t o io . or i io I X, Y, Z ir tio d I $a <_d b$ or t o ri - poi t a b i $I(a) < I(b)$ d i po iti or $I(b) < I(a)$ d i ti . k- ro t vw i r pr t t li t (v b , b , b , $\ldots$, b_k w) r b , b , $\ldots$, b_k r t o vw. o t t o ti i r t o t o oor i t t r r o r t it i r t t ( ) b_i $- b_i$ i k i i -p r ll l tor (2) b_i $- b_i$ i i i r t ir tio to $b_i - b_{i-}$ i k. l t o tor • i ot • .

i tio ri t l orit i to o tr t - ort o o l r i p rti l r p t o t i pl t tio o t l orit i r p rti t to o r p ri t. or o t l orit t t or r o l i t r t i t li i ptoti or t- o or t ir p r or- ro o io i pro t to t l orit i i pr ti i o t t- tor i pro t i o t ti rit ri . r r po i l o r i pl t tio i l i pro t . or pl r o ri -pl ot o t i i rt or ro i r i t r i it ol t l t o .

o ri t COMPACT il o l o- rit to] i i r l t to t ir i pl t tio . COMPACT-7 l orit po itio t rti i $O(\ \overline{n})$ $O(\ \overline{n})$ ri i t (Z)-pl pro r i it $O(\ \overline{n})$ $O(\ \overline{n})$ $O(\ \overline{n})$ ol t o t p r .

riti l o po t o t i pl t tio o t l orit i t o tr - tio o r p H o rti orr po to t to ro t o

t $(Z$ $)$-pl i il rl or ro t lo t $(Z$ $)$-pl .
] rti r t i H i t orr po i t rt i t ro
or i t ol . rt - olo ri o H t r i t i t t
i r ro t . i r t t ro t t t i t o
ot i t r t. o r i pl t tio rti r t i H i t orr po -
i ro t ill i t r t i ro t t t i t. r l t r r
l i H i t i ppro i pr ti l olo r t r or
l ol i . t q ti l r l orit to rt olo r t
r p H. ot t t t i t o ill i pr ti l olo r t t t o
o i l] i ril i i r t olo r to i
t rt i t ro or i t ol i t ill ot i t r t
i ro t t t i t.

l o i pl t t Compact-6 Compact-5 ri tio o t
Compact l orit i pro r i it t o t i
p r r p ti l it $O(\ \overline{n})$ $O(\ \overline{n})$ $O(n)$ $O(\ \overline{n})$ $O(n)$ $O(n)$
ol r p ti l . ri o or t Compact-7 l orit i
plo olo ri t o to t r i t i t o .

\- ort o o l r p r -
i i i to i ••••••••••••••• i o t o rti r i t ri -pl .
i pl t t 3-Bends l orit o •• • ••] t DLM
(i o l o t pl o t) l orit o oo] ot o i pro-
r l po itio r i it $O(n\ )$ ol . r i pro t
3-Bends l orit t o t t r p r . r i pro
t DLM l orit t o t o r p r r o t o t
2− p r . or r p it i r t o t t DLM
l orit pro r i it t o p r .

i pl t t Incremental l orit
o p o t olli 9] t Reduce-Forks l orit o i tti t
•• • ••]. Incremental l orit i pport t o -li i rtio o
rti pro r i it $O(n\ )$ ol t o t t r p r
. o o o t ol or t r o t li
or t Reduce-Forks l orit . ot o t l orit i ol r
o r itr r i io t t r i pro i r ro o i pl-
t tio to ot r.

ot t t to ti o tr i t ot i pl t r o
l orit i oo] or t Dynamic l orit o lo o •• • •• 2] i
pro 6- r i it $O(n\ )$ ol pport t o -li i rtio
l tio o rti .

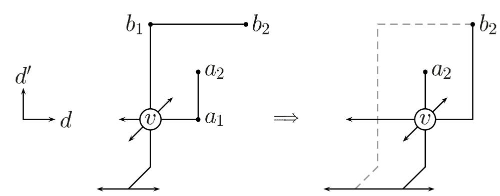
i tio ri r o t iq or t lo l r t o - or-
t o o l r p r i . r t i ppli to r itr r
r i i i t i pro i t l t o o t t ti rit ri l t
o i o ol r o or t l t o .

MoveVertex r t tt pt to r o ro i r - i o i rt v to t r t o ro t i i t to v. ppli to r i $D(G)$ MoveVertex(v d) i ppli or rt v $V(G)$ ir tio d X, Y, Z .

MoveVertex(rt v ir tio d)

---

t (v a , a , a , . . .) t ro t i i t d-port v. t d t ir tio o t t (a , a). t r i o ro t t v i t d port or i t r i ro t (v b , b , b , . . .) a $<_d$ b t lo oi o o ot r t ro t i t r tio o v to a r ro t t i i t to v ill tr t i i . .



• •• • • • MoveVertex r t.

PermutePorts r t tt pt to r o ro i r i r i i t port t i rt to it i i t . ppli to r i $D(G)$ PermutePorts i ppli to rt v $V(G)$.

PermutePorts (rt v)

---

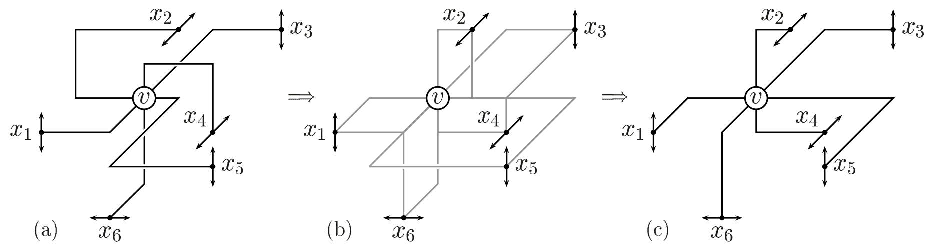
t vw , vw , . . . , vw_d t ro t i i t to v r d (v). plit ro t vw_i i d i to o po t vx_i $x_i w_i$ r vx_i i t i l ro t tir l o t i i ri -pl l o o t i i v i i . 2( ). t S vx_i i d . to S - or - ro t ro v to x_i i d i o ot i t r t t r i r o t r p i i . 2( ). i d p ir i o -i t r ti ro t i S o or vx_i it t i i tot l r o . (ro t t i t i t ori i l ro t r i S.) o t t o t ro t it t ppropri t $x_i w$ i i . 2( ).

RemoveSegment r t i to r o r o i i t i ro t . ppli to r i $D(G)$ RemoveSegment i ppli or vw $E(G)$ p ir o p r ll l t (b_i, b_i) (b_j, b_j) i vw.

RemoveSegment(vw t (b_i, b_i) t (b_j, b_j))
•• • • • • • • • •••• • • (b_i, b_i) (b_j, b_j) r p r ll l t o vw (b , b , . . . , b_k) t t $j > i$.

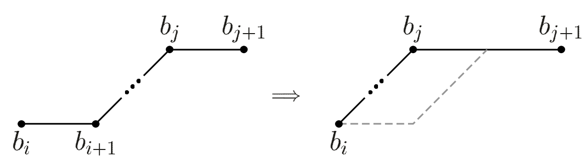
---

t • t tor b_i $-$ b_i. o i r t ro t (b , b , . . . , b_i, b_i $-$



• •• • • • p o t PermutePorts r t.

•, b_i − •, . . . , b_j − •, b_j , . . . , b_k) t t i b_i i r o ll ri poi t ro b_i to b_j r tr l t −•. t i ro t o ot i t r t t r i r o t r i t r pl vw t i ro t r o l-i t r tio r t ro vw.



• •• • • • RemoveSegment r t.

CombinePlanes r t i to r t ol o i r- i o i i t pl . ppli to r i $D(G)$ CombinePlanes i ppli to ri -pl i t r i .

CombinePlanes(i io I i t r x)

---

t (I x)-pl t (I x +)-pl o i it o t i li or rt i t r tio t o o.

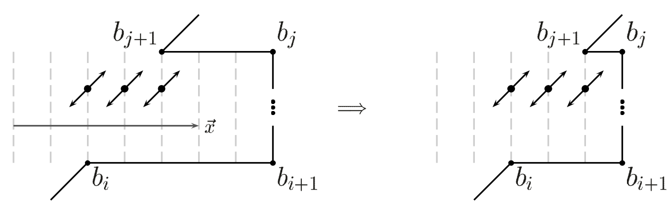
ShortenU-Turn r t i i o t i il r to t RemoveSegment r t i to r t l t o i ort i p r ll l t i t ro t . ppli to r i $D(G)$ ShortenU-Turn(vw, (b_i, b_i), (b_j, b_j)) i ppli or vw $E(G)$ p ir o p r ll l t (b_i, b_i) (b_j, b_j) i vw ti i t i p t o itio .

ShortenU-Turn(vw t (b_i, b_i) t (b_j, b_j))
•• • • • • • • • • •• • • • (b_i, b_i) (b_j, b_j) r p r ll l t o vw (b , b , . . . , b_k) t t $j > i$ t tor • b_i − b_i • b_j − b_j r i t ir tio .

---

o i r t ro t (b , b , . . . , b_i, b_i − •, . . . , b_j − •, b_j , . . . , b_k) (it r- t l o r o) r • i tor poi ti i t ir tio • it l t i t r , , . . . , • + • − pl t ro t

vw it t ort t o ro t (t it t l t) t t o ot i t r t t r i r o t r i . o l -i t r tio i vw.



p o t ShortenU-Turn r t.

r l r t ll DrawTrees&Chains i i r t i t r to t pr io r t . t o i t o t o p . t r t p rt i r-ti r r o rt i p t r r pl i l . t o p i i i to o r t r ot r r t ppli t r o rti r r i rt t ro t or t i l r r-pl p t (op ll) it r . o ri t r t p o t r t.

RemoveTrees&Chains(r i $D(G)$)

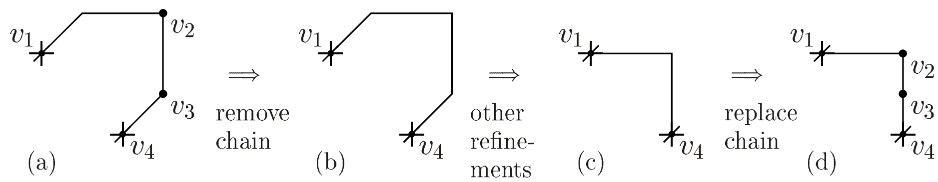
. p t l r o rti it r o (i t rr t r i) t t i r o ' tt tr ro t r p .
2. i i l p t $(v v , v v , \ldots, v_{k-} v_k)$ r r rt v_i pt po i l or v v_k r t o. pl i $(v v , v v , \ldots, v_{k-} v_k)$ t $v v_k$ i i . ( ).

o p o t r t i ollo .

RedrawTrees&Chains(r i $D(G)$)

. rt rt v r o RemoveTrees&Chains i t oppo it or r to t ir r o l ollo . t w t t rt to v. oo r port t w i t t vw ro t it-l t t. port i t t ro t vw it-l t t. t r i oo r itr r r port t w i rt pl t to w t t vw ro t it-l t t.
2. or i $(v v , v v , \ldots, v_{k-} v_k)$ r pl t ro t $v v_k$ i t RemoveTrees&Chains p pl t rti $v , v , \ldots, v_{k-}$ t t o t ro t $v v_k$ l p po i l . t r r or rti t t po itio t r i i rti r itr ril o t ro t i . ( ). t ro t l ri poi t t rti t i rt pl to o o t t rti .

o ri o ll t r t r o i i to o l orit i ll 3D-Refine. i port t i io to i t or r



• •• • • • p o t RedrawTrees&Chains r t.

o ppli tio o t i i i l r t . o t r i opti l or r r i r t o i tio o t r t o 6 o t • •••••• r i (t.). r i r t r i t Compact-7 3-Bends Incremental Reduce-Forks l orit ppli to 9 r p it n ,2 , . . . , 9 rti . r p r t i pro t i r o i o i l t i r t o t % ro t or ri o t r t t r p r t i pro t i r p r i r t o t % ro t or ri o t r t . o l t t t or ri o t r t i ot ' i i t . r itr r or ri o t r t i o i pr t i t ollo i l orit .

3D-Refine(r i $D(G)$)

RemoveTrees&Chains($D(G)$)

MoveVertex($D(G)$) PermutePorts($D(G)$)
RemoveSegment($D(G)$) CombinePlanes($D(G)$)
ShortenU-Turn($D(G)$)
o
RedrawTrees&Chains($D(G)$)

o l r t i po i l i r o ol tot l r o or tot l l t i MoveVertex i i r t tot l l t . r or t l orit 3D-Refine ill o ti til t ol t r o ot r rt r ill t o l r t tot l l t t l orit 3D-Refine ill t r i t .

r t t o i p t r i i r t o r t o t l orit ppli to t r p i t p ri t o i t-ti t •• • ••]. r o l r t i pl o t r p r r o r. t or r t t i pr ti l r p r i ppli tio it i l to r p it r t r it t r r o r. ll t r p t t o r i pro t l orit ppli to t r p t • •••••• r p r i . r r 2 r p it n rti or n ,6, . . . , . t r r 92 r p r i .

o to i p t r i i o i to r o l r t i pl o t r p i r ‘ l o t 6-r l r. ot t t t i i t r t p ri t ri t p r or o - ort o o l r p r i l o- rit o i r r p . r tor r o ti to r o to t r p til t r r o p ir o o - t rti ot it r l t i . i 2 r p or n , 6, . . . , rti- . ll t r p t t o r i pro t l orit ppli to t r p t r p r i . o l p t t t 6-r l r r p r t o t i lt to r ll port t .

o r port t t o r t o t i p t r i . or o t r i r p t l t i i i l r t til it ot ppli to portio o t r i . i pro i r t to t r i t i r t ppli r t ol o t r i . - l r port t p r t i pro t o t t ti rit ri r i i i l r t r t 3D-Refine l orit r o r t r i o r t r i .

r pro t or r t.

t				s		6			s	
	o .	.	.	. .	. .	o .	.	.	. .	. .
MoveVertex	3		.	2		.	.	.	2.3	2.
PermutePorts	.	.	.	.3	.	.	3.	.	.	.
RemoveSegment	2		.3		9.3		3	.2	.9	.
CombinePlanes		.	.	3	32		.9	.3	23	2
ShortenU-Turn	3		2.	2	2	3	.	.		
3D-Refine										

r lt i l o t t t r t r o i r l or - ti ppli to t r i t to t r i . i i p t i t r p r r o r r t r p i r t r or t r port t r- t llo t r t to ppli or o t . r p or rpri i o r tio i t t t CombinePlanes r t i t o t l o t r t i t r o r i t ol t l t o .

o r port t i pro - t i t t ti rit ri i ppl i t 3D-Refine l orit to t i p t r i . r tl o i r t r o . or o t o t l orit t r o (ot r i) p r i o i- t t ro ll l o n or t Reduce-Forks l orit o t t

t r o r ll i r it n. t r or ri o r r-lt i t ollo i r. l 2 r port ( ) t r i r o p r i r i pro l orit (r o r t r p t r p) (2) t r - i r o p r i r i o t i t r ppl i t 3D-REFINE l orit to t r i pro l orit (r o r t r p t r p) ( ) t p r t i pro t i t r i r o p r i ppl i t 3D-REFINE l orit to r i pro l orit (r o r t r p t r p). ot t t ( ) i ot i pl t p r t i pro t i (2) ro ( ).

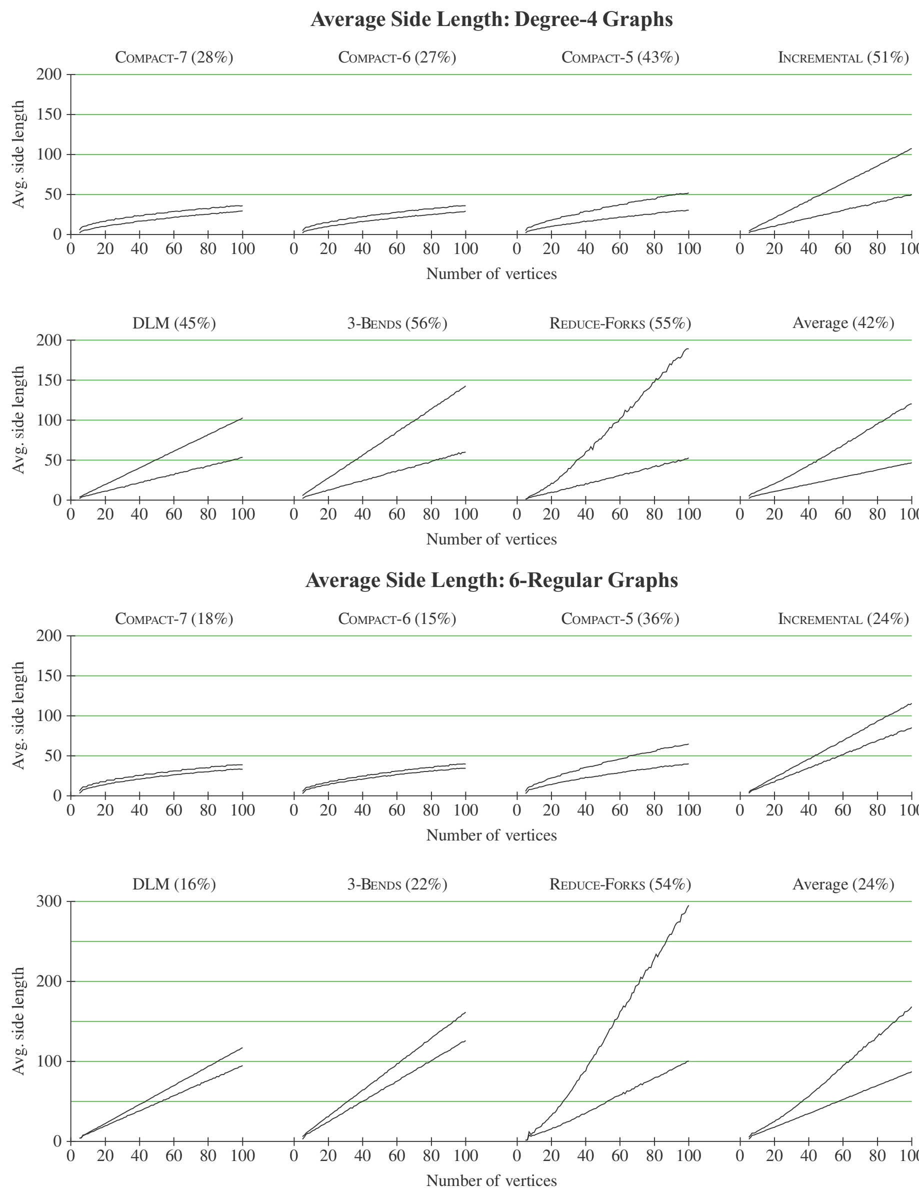
pro t t u r o or t r 3D-REFINE.

		s	6	s
	.	.	.	.
COMPACT-7	. →2. 2 %)	. → . 9 %)	. →3.3 32 %)	. → . . %)
COMPACT-6	.3→2.3 %)	. → . %)	.2→3.3 22 %)	. → .9 2. %)
COMPACT-5	3. →2.3 3 %)	. → .9 .3 %)	3. →3. 9. %)	. → . .2 %)
3-BENDS	2. → . 3 %)	3. →3. .2 %)	2. →2. .2 %)	3. →3. . %)
DLM	2. → . 2 %)	3. →3. . %)	2.2→2. . %)	3. →3. . %)
INCREMENTAL	2. → . 2 %)	3. →3. . %)	2.2→2. 3. %)	3. →3. . %)
REDUCE-FORKS	.2→2. %)	→ . %)	. →3. 33 %)	→ .3 3 %)
r	3. →2. %)	.3→ . %)	3. →2. %)	. → . %)

r lt i l 2 o t t t 3D-REFINE l orit o i r l r t r r o i ll o t r i to l r t t i t r i . r i i i t i pro t i t i r o p r o l i t r i it r l ti l prior to t ppli tio o t r t l orit .

i . 6 i . 7 pr t or l orit t r i l t o t o i o t r l t or t r t ppli tio o t 3D-REFINE l orit (r o r t t r p or l o n). rt t t or t r i l t o t o i o r t r t t o i o ol or o pr t tio o o r t r i l t i t root o t ol . r lt or t i l t o itt t lo l r l t lo o r lt or t r l t . or l orit i . 6 i . 7 l o o (t to t l orit) t r p r t i pro t o r ll r p . i pro i r o t pti ilit o t r i pro t t l orit to i pro t t 3D-REFINE l orit .

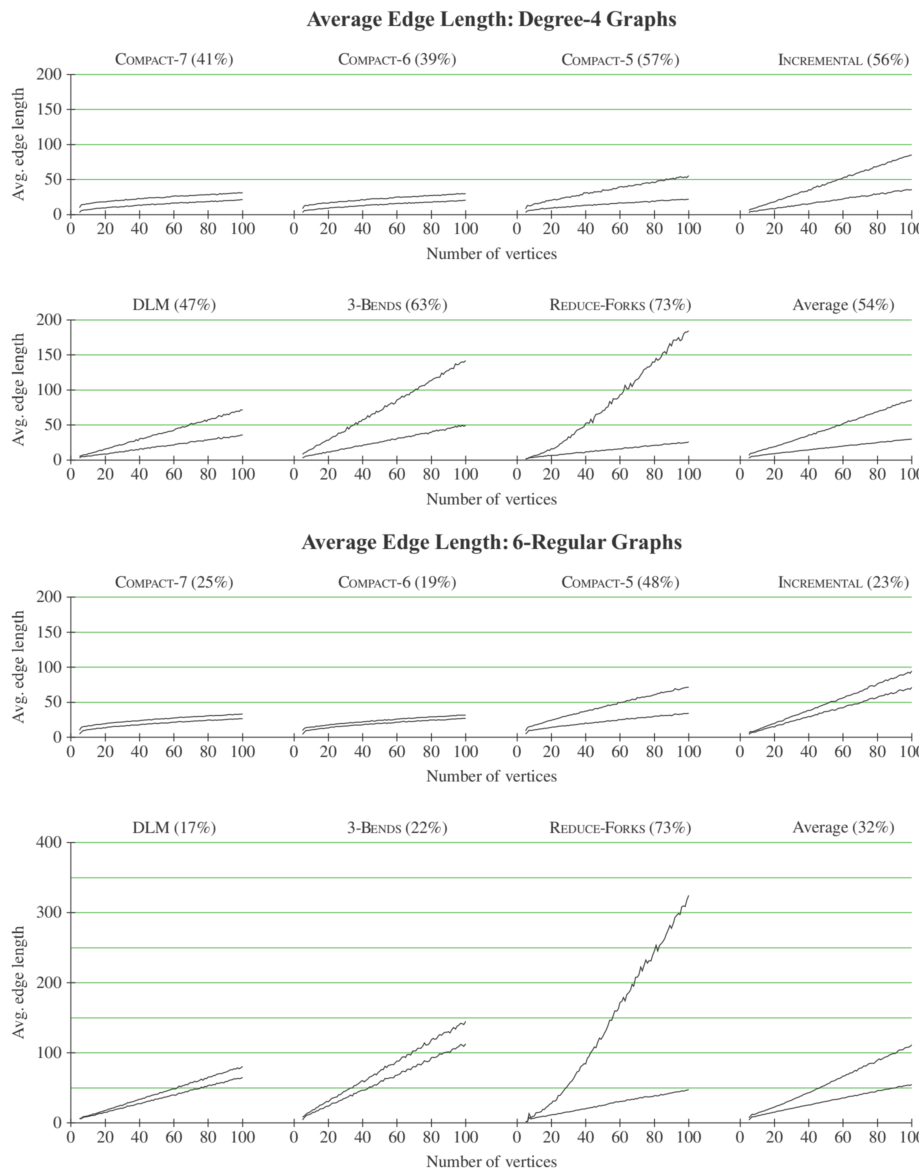
t it t r o t r lt i i . 6 i . 7 o t t t 3D-REFINE l orit o i r l r t o i o ol t l t o i ll o t r i to l r



• •• • • • r ou o t or t r 3D-Refine.

t t i t ••• •••••• r i . ot t t t r i pro t o 2% or t r o i o i l t o t i or t • ••••••• r i orr po to i pro t o % i t o i o ol .

o o p r t p r or o t r - i l orit r tl it o t r t t ollo i t ppli tio o t 3D-Refine l orit l 2 i . 6 i . 7. r r lt r ll



• •• • • • r t or t r 3D-Refine.

o r t or t pp r o t li or l orit o- r t r lt i] or t • ••••••• r p . p r or o t l orit o t ••• ••• •• • r p o ti t tr o r or t • ••••••• r p it o r ptio . Reduce-Forks l orit p r or oti l or o t ••• ••• •• • r p r l ti to t ot r l orit o p r it t • ••••••• r p .

t r o ol l t t t l orit (it o t r - t) r COMPACT7 COMPACT6 COMPACT . ot t t t or t-ol o or COMPACT7 COMPACT6 r $O(n^{\ /\ })$ $O(n\ )$ r p -ti l . t i pr ti t i r i t ir p r or i li i l i to t olo ri t o i i t. 2. DLM INCREMENTAL r t t t p r or i l orit ollo 3-BENDS REDUCE-FORKS. i il r p tt r r o p ri t l orit t r r - t pt t t REDUCE-FORKS p r or l o t ll t COMPACT l orit t t i t r i pro t REDUCE-FORKS l orit r i l pti l to i pro t 3D-REFINE.

t i p p r ri r o po t-pro i t iq or t r t o - ort o o l r p r i . r i l orit t ti l i pro t to ll o t t ti rit ri r p i ll ppli to r l ti l lo r r p . i p ri t o tri t to t o oi r r ort to - ort o o l r p r i or ppropri t or i li tio p rpo . i port t t r t p to r t i o l i t lop t o i t t tr t r or t i pl t tio o l orit r t or - ort o o l r p r i .

to i or t i l pport to ri io tri i or i l ppl i t o ro t i pl - t tio o t INCREMENTAL REDUCE-FORKS l orit .

. . . . ro - o or pro t t qu o o oro r . port -2 - 3 r t o t r oo 2 .

2. . o o . rt or . o . . t . u 3- o ort o o r p r . r to p 9 .

3. . tt t . tr . r u. p t pu ppro to 3 ort o o r . t 2 p .

. . . t rk . t . t qu o o o oro r or t r o ort o o r p r . *tt* 2) 9 3 99 .

. . . o . t . r o ort o o r p r or t . *s t* *t* 3 2 .

. . ö r . . u . pro ort o o r t *M*- or t . t 2 p 2 3 .

. . . o o oro . . r . t r to o t 3- o p . *s* *t s* 2 2 r 9 .

. . r to tor. *t t t* o u 3 o *t* *t s* *t* pr r 999.

9. . p ko t . . o . or t or r t ort o o r p r - t r o . *t s* 3) 999.

. . tr . r u. 3 u too or t r o r p r .
. tt t tor *t t t*
o u 3 3 o *t t s t* p 2 29 pr r 99 .
. . . . . kou . . o . t o ort o o r p r -
. t 2 p 3 2 3 .
2. . t tor. *6t t t*
o u o *t t s t* pr r 99 .
3. . . oo . or t or t r - o ort o o r p r .
t 2 p 332 3 .
. . . oo . *s t* . t oo o
o put r o t r r o r t u tr 2 .

ll r q t

i sr stitut i 32 sr
barequet@cs.technion.ac.il
p http://www.cs.technion.ac.il/~barequet

dr i r p i t p is i v r v rt t r p is t d t t r i s strip idt i d s t t i t r v rt t r p . sti t t i u p ssi v u ( ) -s r i t - di t dr i r p it v rti s i is t i d i t u it squ r . s r u d d upp r u d ( ) ( ) $\Omega($) d ( ) ($^{\cdot/\cdot-\varepsilon}$) r r itr ri s . i pr v t r it r u d i s rr t t - us i r 's tri pr .

tri pti i ti i r 's tri pr .

t s p p r r pr s t r p - r pr l s pt t pr l t r l tr st l s l r pp r r t s l t t s pr l . r "s-t p " pr l s ttr t tt t t r t t l st t r . t r st r r s r rr t t r l t r - t r t r l tr s . . 2 3 6]. p ll l l t r r p t t t sq r s t t rt t r p s - r str p ss t t t rt s r str p ss t t t r rt . r l s t t t t str ps. s str p t t s r t r t r p r t rs

. p t t t s r l t l s t t rl t str p.
2. r t s t r t t t str p.
3. t t str p.

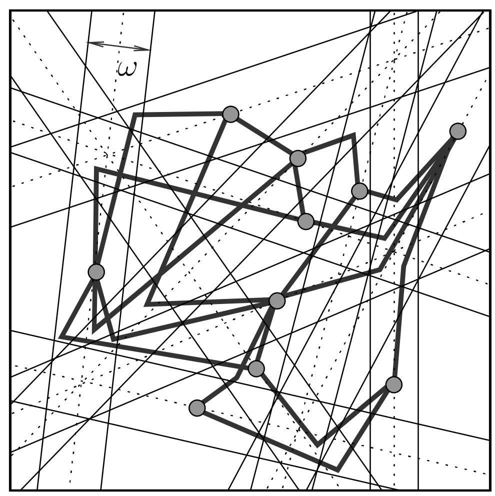
s s r l ts s s r p ts r l t t t sq r . l ll t s r l ts s t pr p rt t t s r l t s r p t s r t r s r l t. r t r st t ll pr l

l ll t s r l ts t t t sq r t s t p ss l t ( ) t s r l ts?

○

q t t ( ) s t p ss l l r

r t t t sq r r p t rt s. t s p ss l t s p r p s s r p t str ps t s t t r str p t s rt ts t rl t t s t t t r rt t r p s r . s t r p t r r r t r l



-s r i t di t r p dr i

t t s r l ts (t t sq r) t r q r t s s t t pr l r r. -s r l t t r s s t ll t l pp s r l ts t . s s r t t s tl t s r l ts r t r s r p ts r str p ts t r rt t st pt t rs t (t t t sq r) t t str p ss t t t rt . r s l s r t pr l l tr p s r t rt s r p rts t s r r s t p r ll t r s p ss t r l s t rs r r t r s ss t t p rt. (s s s t s l r t r l t t t t s s r p - r l r t s s . .].) r l r pp r s r t s r l ts pr l r ts r l t t t s l r s tr l pr l 7]

p ts t t sq r t s ( ) t p ss l r t tr l s t r t s p ts?

l r p s t tr l pr l t rs s t t tr t t s t st t t s. t t r s st ll l r p t t st rr tl - l r pp r s r ( )

Ω(l)] (/ $-\varepsilon$) (r)]. pr s s r
t st r t s pr l (l t r s lts l s t l.) s
t 9]. r l t t t tr l pr l t s r l ts
pr l p s s s l r p t t s s t s p p r r
t l tt r pr l . pr t r t l r r pp r r t
pr l s ll ls pr t r sp t r t t r pr l .
pr l s ss t s p p r s ls t t t ll .
pp s t t t t l l ( ) s . s r t s t t t r l s .
r p r p ts s str p t ( ) t s
t r p ts r ( ) s t st t . s t
t t r str p" s rs l pr p rt l t ( ).
t s] t t ( ) ($\frac{}{n\cdots}$). t s r t st t
t ' ' str ps r t r l t t t r t t
t r p ts t t . l l t str t r t t
t str p- t t (s t st s t p ts) l r s
tr l pr l s r. t t s t t r l t t l-
r s pr l t t s r l ts pr l r ll t s r l ts r str ps
t s t .

• • ••• • ••••••• •• $\omega\bullet n\bullet$

rst st l s s l r pp r s ( ).

•• • •• •• • ••••

• •••••• •• ( ) Ω( )•

•••••• st l s t l r r s pl pl . t rt l
s r l ts t s t t t r t r rs t t rs t. s r
p ts l t r l t s r l t t rl s sl
s r s r t r s r l t.

•• • •••• ••••

• •••••• •• ( ) ($\overline{}$)•

•••••• t $\overline{\quad -\quad}$. rt t t t sq r t ll r s ll
sq r s s l t . l t s r l ts t t sq r .
t r t s t st ($-$) s ll sq r s t r st st
s sq r t t t s t s r l t s r p ts. st t
t s t s r s s t st $\overline{2}$ t r r t s r l t t t
2 $\overline{2}$ 2 $\overline{2}$ $\overline{\quad -\quad}$ ($\overline{}$).

• tu s t . s d i t is rk t t $\mathcal{H}$( ) ($c\sqrt{\cdots n}$ •/•) r s st t .

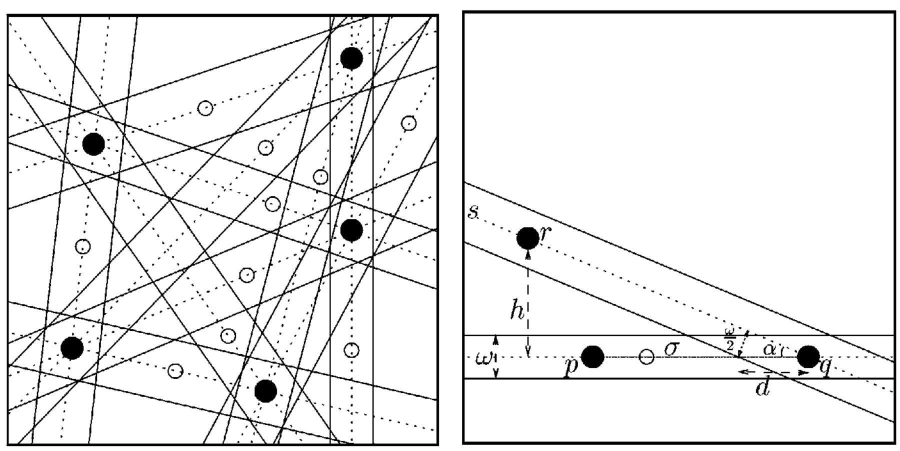
• t it is r t r s s us r r t d t i pr v (i r s) t st t pr p rti it i t is u d.

ωn

t s s t pr t s ( ) t t pr s s t . t st t t r l t t l r s tr l pr l t s r l ts pr l .

( ) ( ) ( ) ( ) ( ) ($\sqrt{2}$) ( ) $\sqrt{(2\ )}$

rst s t l r ( ). p t p ts t t sq r s t t t r t s ll st tr l s t r t s p ts ss s ts ( ). (r 2( ) l r s p ts pp r



( ) stru ti ( ) v idi s ur p i ts

ti i r 's p i ts (•) it s r i ts

s l r l s.) t r t $(^N$ l s t p ts. s ll s r l t t rl s t l t s t s r p ts (s s t r l s r 2( )) ll l t r. r s l s spl t t t r l s t ( ) s ts. s l t t s s ts (r t p ts t s t) l t t s r p t s r l t t (s r 2( )). (r p t t s r ll t ( ) l s.) r s p s t ss t t l ss r l t t t s r t l. r t t l t sts r l t t p r s r t r t - p t . s s t str p t t rl ps t st t r t s ll s r l ts s t rl p ss s t r . (r t st s r l t t p r s r ls rl p s t p rt t r t s s ppl t s l s s r s r l ts t t t r s r .) t t t r p t t t r t t

str p . t t l ss r l t ss t t s . t ss p-
t t r t tr l t p ts s t l st ( ).
s t lt t r l t t s t s s 2 ( ) t t s

$$2 \quad ( ) \qquad ( )$$

p t t t rl p t . sl $\overline{2}$
s (2). s

$$\frac{}{2} \quad \frac{\overline{2}}{2} \qquad (2)$$

s st t t q t ( ) q t (2) t

$$\frac{\overline{2}}{( )} \qquad (3)$$

s l r l s s s s t s pp r t l t t p rt
rl pp t r t st (r l t st) s r l t t p r s
r . r r t t t l l t t t rl pp p rt s s t st
$\overline{2}$ (2 ( )). r r t pr l s r l t s r p t
st $\overline{2}$ (2 ( )) t t s

$$\overline{2} \quad ( ) \qquad ( )$$

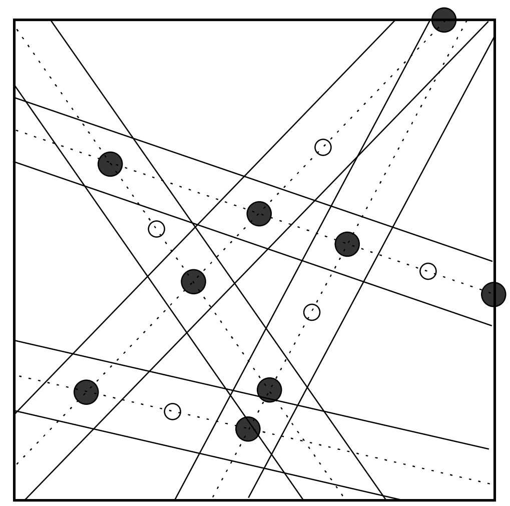
ll s t $\overline{2}$ r r t (+ ( )) s r l ts t
$\overline{2}$ ($\overline{2}$). tt $\overline{2}$ pl t s t r t.

r t t pp s t r t s s t pp r ( ). r
t r 3. r p t s r l ts t p ss l t s
t t t r s r p ts r l t t t sq r . t rl s t
s r l ts t rs t s t t sq r t st (^{n} p ts. -
t rl p s t t l r p ts" t t t t rs t p ts
t s t t s r p t t r sp t s r l t l s.
(r t s p rp s ls s r t t rs t s t s r l t t rl s
t t t sq r .) t t l t s r 2 p ts.

r t t s ll st- r tr l s tr pl t s
p ts (s r 2( )). r s t p ts r t s r-
l t s t rs t t t s r l t t s . t
ss pt t r t s tr l s t st ( ).
t s t
s p ts r . lt t r l t t st s t s
2 ( ) t t s

$$\frac{2 \quad ( )}{} \qquad ( )$$

* i ur 3 is is di i t s s t t t is p t i s t r r r i r
i r p i ts i vi us d tri r . t v r t t
us r ••••• u d t r i r 's tri s.



ti i r p i ts s r i ts

p t t t rl p t . s (2). s t s t t t t 2 (t r s t s r p t t s r l t l r t s r l t t r t r t r t). s

$$\frac{}{2} \quad \frac{}{} \tag{6}$$

s st t t q t ( ) q t (6) t

$$\frac{}{(\ )} \tag{7}$$

s l r l s s ppl s r t l t s . r r t t t l l t t t rl pp p rt s s t l st (( )). t t r t s t t l s t st (s t s s r l t s r p t) s st (( )) t t s

$$2 \frac{}{} \quad \frac{}{(\ )} \quad 2 \frac{}{} \quad \frac{}{(2\ )} \tag{\ }$$

tt 2 — pl t s t r t.

t q t t st rr tl - s r l r s tr l pr - l

( ) Ω(l)

. t t di r t t is r u d | | d t upp r u d $\sqrt{2}$ us d v .

( ) ($/ -\varepsilon$)

ll r s 3 t t r r s lt

( ) Ω(l) ( ) ($/ -\varepsilon$)

ls t pp s t p

( ) ( ) ( )
( ) ( ) (2)
( ) (2)

rst s t t t r sts st t r ( ) (2). ss t t tr r t t s st t sts t t s r r s tl -l r l s ( ) (2). r 3 t r sts st t r ( ) ((2) 2) ( ) s tr t .

l rl s t t t r sts st t r ( ) (2). ss t t tr r t t s st t sts t t r r s tl -l r l s ( ) (2). r 3 t r sts st t r ( ) (2(2)) ( ) s ls tr t .

r 7 pl s t t pr t t l r r t pp r r t s r l ts pr l ll ls rr t l r s tr l pr l s t t tr t s p l t t s t r t t l st l t r .

. . . u . . dri . . ur v . . r i p r r- p s it ir u r r s. r . t r p r i r r u 26. tur t s i put r i 3 pri r- r
2. d . . ut k . k vit . k . (ds.) is r t tr d v it . . rk d. i. ( )
3. d i r . ru r . (tr s ti .) i t ri tr i t . t i rt d i st rk 6
. s . i t . r di . i r 's tri pr . . d t ti i t (2) ( ) 3 3 6
. s . i t . r di . r u d r i r 's pr . . d t ti i t (2) (2) 3 2
6. s r . . s r pr s i is r t tr . i r p d t s
. t . . pr i r . r . d t ti i t ( ) 2
. t . . pr i r . r . d t ti i t (3) (2) 3
. t . . v p ts i i r 's tri pr . dv s i t - ti s (6) 36 3

nos n ot

• ou nt n titut n un i n c o ci nc
• c u tt n titut o c no o n un i n c o ci nc

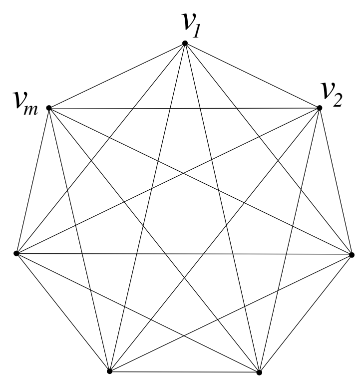
• • ••••••• t r i wn in t n o t t
it tic nt oint n it nt
o n cu conn ctin t co on in oint wit t o t
t t n two cu t o t on oint in co on. n two
c nonic c o to o o ic co t n o t t
to o o ic co t wit • tic c nonic u o
i t t • o o • w ic on to on o t c . o
ow t t co t to o o ic wit • tic non
c o in u i o o ic to n t wit t o t • o •/• •
tic .

G is gr p r wn in t p n or n ur s n two o w i t ost on point in o on. t is it is n s p ir $V(G), E(G)$ w r $V(G)$ is s t o points in t p n n $E(G)$ is s t o si p ontinuous r s onn ting t so t t t s tis t o owing on itions

. no r p ss s t roug n ot r nt o $V(G)$ i r nt ro its n points;
2. n two r s t ost on point in o on w i is it r o on n point or prop r rossing.

$V(G)$ n $E(G)$ r t n o G r sp ti . s t t H is *(* o G i $V(H) \subseteq V(G)$ n $E(H) \subseteq E(G)$. wo topo o-gi gr p s G n H r i t r is n in i n pr - s r ing on -to-on orr spon n tw n $V(G), E(G)$ n $V(H), E(H)$ su t t two g s o G int rs t i n on i t orr spon ing g s o H o (s]). g s o topo ogi gr p r str ig t- in s g nts t n it is . g o tri gr p w os rti s r in on position is . ious n two o p t on g o tri gr p s wit m rti s r w k iso orp i to ot r n to t on g o - tri gr p C_m w os g s t onsists o si s n or s o r gu r m-gon. (ig. .)

s
○



C_m

ir t nsi it r tur on topo ogi gr p s o us s on r w sp - i qu stions n t r is no st n r t r ino og . or topo ogi gr p s r os n u 73] (s so]) us t t r goo r wings w i ron u r ort ng rs n n ür nn 9] 7] 9] 9] si p t r wings. or o p t topo ogi gr p ing 6] n ng rs n 7] us t t r i rsion. ost popu r pro s in t is r ur n's ri k tor ro 77] (r nki wi 's onj tur 69] n ot r pro s out i. . out t nu r o rossings in rt in r wings o gr p 9]) n onw 's r k onj - tur 7] 97]] (n ot r pro s out t nu r o rossings in rt in r wings o gr p 92]).

s st ti stu o g o tri gr p s w s initi t r os it n ni 66] upit 79] n r s. (99] n 9] pt r or t ost r nt sur s on t su j t.) t is not r to s t t r K_n o n rti s s non- rossing su gr p iso orp i to n tri ngu tion o o ngt n (. 9]). ons qu nt K_n s non- rossing su tr iso orp i to n tr o n rti s. n p rti u r K_n s non- rossing p t o n rti s n non- rossing t ing o si $n/2$. n t ot r n it is known t t K_n s t st onst nt ti s $\overline{n}$ p irwis rossing g s.

ur i is to st is n ogous r su ts or topo ogi gr p s.

n

T c og $^{/6}$ n

c og $^{/6}$ n

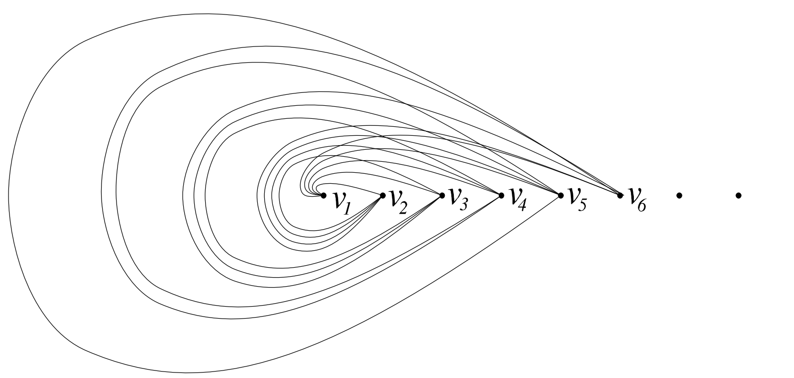
or ing to w known t or o r os n k r s 3] 6] n s t o n points in g n r position in t p n ont ins su s t wit t st c og n nts w i or t rt s t o on po gon. (roug out t is not t tt r c pp ring in i r nt ss rtions not unr t positi onst nts. st known oun in t st st t nt is u ot n tr 9].) r os- k r s or n r or u t s o ows.

n

C_m

m c og n

situ tion is or o p i t or gr p s. n t ir stu o topo ogi o p t gr p s wit m rti s n wit t possi nu r $\binom{m}{}$ o g rossings r ort n ng rs n 92] oun r-wing w i ont ins no su gr p w k iso orp i to C . t is r wing pi t in igur 2 n not it T_m.



T_m

s ow t t on nnot oi C_m n T_m in su i nt rg o-p t topo ogi gr p .

n

m c og og n

C_m T_m

or w turn to t proo o or 2 w r p r s t nitions o on n twist o p t topo ogi gr p s.

t K_m o p t topo ogi gr p on m rti s. t r is n nu r tion o t rti s u , u , . . . , u_m su t t

(i) two g s u_iu_j $(i < j)$ n u_ku_l $(k < l)$ ross ot r i n on i $i < k < j < l$ or $k < i < l < j$ t n K_m is ;

(ii) two g s u_iu_j $(i < j)$ n u_ku_l $(k < l)$ ross ot r i n on i $i < k < l < j$ or $k < i < j < l$ t n K_m is .

t K o p t topo ogi gr p wit $n +$ rti s. g s o K i i t p n into s r s pr is on o w i is un oun . it out oss o g n r it w n ssu t t t r is rt v $V(K)$ on

t oun r o t un oun . t rwis w n pp st r ogr p i proj tion to tr ns or K into r wing on sp r n t n not r proj tion w n turn it into topo ogi gr p w k iso orp i to K w i s tis s t r quir prop rt .

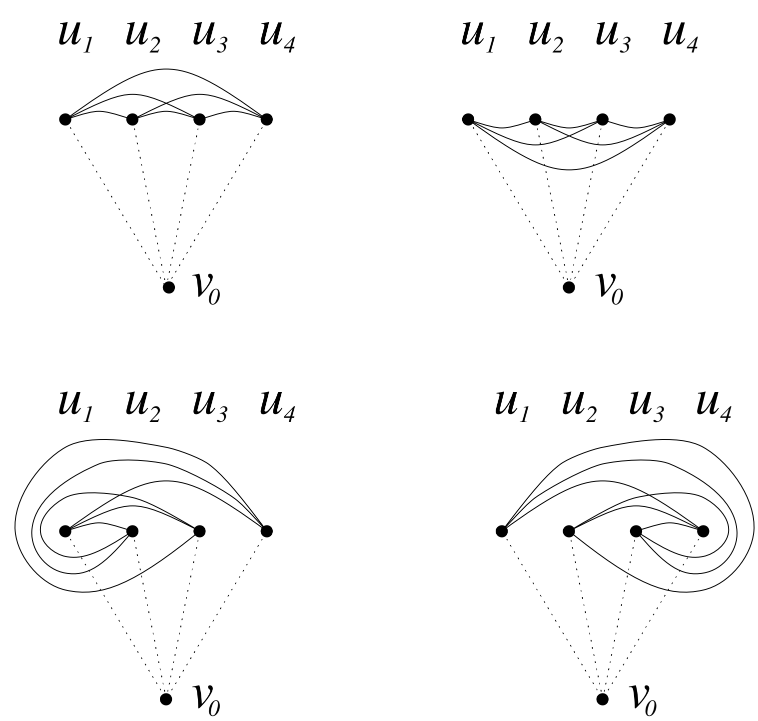
onsi r g s n ting ro v n not t ir ot r n points v , v , . . . , v_n in o kwis or r.

o or t trip s $v_iv_jv_k$ $i < j < k$ n wit ig t i r nt o ors or ing to t o owing ru s. o or is r pr s nt ro-on s qu n abc o ngt 3. or n $i < j < k$

. s t a i t g s v_iv_j, v v_k $E(K)$ o not ross n t a ot rwis ;
2. s t b i t g s v_iv_k, v v_j $E(K)$ o not ross n t b ot rwis ;
3. s t c i t g s v_jv_k, v v_i $E(K)$ o not ross n t c ot rwis .

t is s to s t t t o p t topo ogi su gr p o K in u t rti s v , v_i, v_j, v_k (s n ot r o p t topo ogi gr p wit rti s) s t ost on p ir o rossing g s. r or w

on o t o ors or n o ur.



f f s 's or t r is n m- nt su s - qu n $(u$, u , . . . , $u_m) \subseteq (v$, v , . . . , $v_n)$ m c og og n su t t trip s $u_iu_ju_k$ r o t s o or (c is positi onst nt).

uppos rst t t trip s $u_i u_j u_k$ r o o or (s ig. 3). n two g s $u_i u_j$ $(i < j)$ n $u_k u_l$ $(k < l)$ ross ot r i n on i $i < k < j < l$ or $k < i < l < j$. t is u , u , . . . , u_m in u o p t topo ogi su gr p in K.

o t in t t s rossing p tt rn tw n t g s in u u , u , . . . , u_m i trip s r o o or (s ig. 3).

uppos n t t t trip s in u , u , . . . , u_m r o or . n two g s $u_i u_j$ $(i < j)$ n $u_k u_l$ $(k < l)$ ross ot r i n on i $i < k < l < j$ or $k < i < j < l$. n t is s u , u , . . . , u_m in u o p t topo ogi su gr p .

in t s w n trip s in u , u , . . . , u_m r o o or is isoorp i to t pr ious on un r r fl tion r rsing t ori nt tion o t p n (n n t nu ring o t rti s v_i (i n)). □

t is r s to k t t ot C_m n T_m ont in non- rossing opi s o r tr wit m rti s. us w k r rsion o or wit c og og n inst o c og $^{/6}$ n r i o ows ro or 2. n t n t s tion w pp so w t or i t rgu nt to i pro t is oun .

t G topo ogi o p t gr p wit n $(n+\ )$- nt rt s t V. s t s nu ring v , v , . . . , v_n o t rti s s in t pr ious s tion. or n $< i < j$ w s t t v_i v_j (in not tion v_i v_j). s or o or t trip s $v_i v_j v_k$ ($i < j < k$ n) wit o ors , , , n .

r ists n m- nt su s t U u , u , . . . , u_m v , v , . . . , v_n m $\overline{\text{og}\ (n+\ )}$ su t t t trip s $u_i u_j u_k$ n $u_i u_j u_l$ t s o or or n $i < j < k < l$.

f onstru tion is r ursi . t U v , v n V V v , v . uppos t t or so 2 $p < m$ w r oun two su s ts U_p u , u , . . . , u_p n V_p V wit t prop rti s

. u u u_p,
2. r nt o U_p pr s nts o V_p,
3. V_p $\frac{V. \_. \ -}{:}$.

t u_p t s st nt o V_p wit r sp t to t or ring .' in w us o ors or o oring t trip s t r is su s t W V_p u_p wit W $(V_p\ -\ )/\ ^p$ su t t or i p trip s $u_i u_p$ w (w W) t s o or. t U_p U_p u_p n V_p W. n s o put tion s ows t t t is pro ur n r p t t st $\overline{\text{og}\ (n+\ )}$ ti s. □

n t $u_i u_j$ $(i < j < m)$ s t o or o trip $u_i u_j u_k$ or n $k > j$. t p o $u_i u_m$ n n r itr ri .

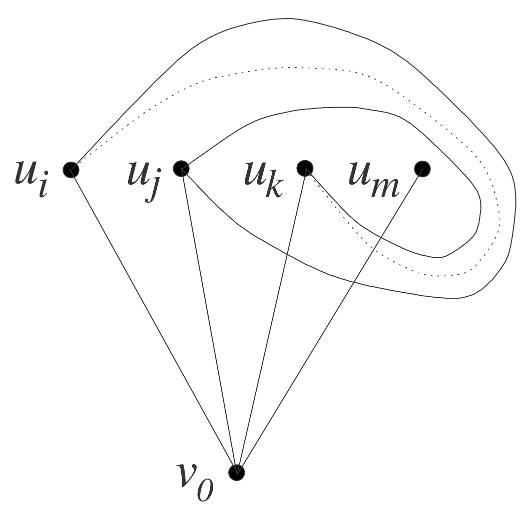
t $G($) n $G($) not t topo ogi su gr p s o G onsisting o g s o t p n r sp. w os ot n points ong to U u , u , . . . , u_m . topo ogi su gr p onsisting o ot r g s o G in u U (o t p s n) is not G .

t $i < j < k < m$.

(i) u_iu_j n u_ju_k ong to $G($) t n so o s u_iu_k.

(ii) u_iu_j n u_iu_k ong to $G($) t n so o s u_ju_k.

f u_iu_j is o t p it ust ross ot v u_k n v u_m. t t p o u_ju_k is so it ust ross v u_m too. sing t ssu ption t t two g s t t s r n n point nnot n ot r point in o on w o t in t t u_iu_k ust ross v u_m w i i p i s t t its t p is so (s ig.). is pro s p rt (i). rt (ii) n st is si i r . □



u_iu_k v u_m

$G($), $G($), or G ont ins o p t su gr p o si r m / t n G s non- rossing su gr p iso orp i to n tr o r rti s.

f uppos t t w w . . . w_r in u o p t (topo ogi) su gr p in $G($). t is s to s t t t is su gr p is i. . it is w k iso orp i to T_r. k n r itr r tr T wit r rti s. t rting t n rt z $V(T)$ p or ot r rti s o T using . t z , z , . . . , z_r nu ring o t nts o $V(T)$ in t or r in w i t r n ount r t gorit . n t ing $f(z_i)$ w_i (i r) ps T into non- rossing op o T in $G($) n w r on .

s w n $G($) ont ins o p t su gr p o si r n tr t si i r .

ssu now t t G s o p t su gr p wit r rti s w w . . . w_r. t is s to s t t i two g s w_iw_j $(i < j)$ n w_kw_l $(k < l)$

ross ot r t n w $i < k < j < l$ or $k < i < l < j$. n ot r wor s i two g s o t is su gr p ross ot r t orr spon ing g s so ross in r wing on t s rt s t w k iso orp i to t r wing C_r. r C_r ont ins non- rossing op o r tr wit r rti s so t s is tru or G . □

n i w o t st i it r ins to pro t t t st on o $G($ $)$, $G($ $)$, n G s o p t su gr p o si r $m^{\,/}$. uppos in or r to o t in ontr i tion t t t is is not t s .

so nt u U u , u , . . . , u_m t st $r-$ rg r n ig ors in $G($ $)$ wit r sp t to t or ring t n i 3.2 (ii) t s n ig ors tog t r wit u wou in u o p t su gr p in $G($ $)$ ontr i tion.

ow w r ursi onstru t s qu n w w . . . onsisting o t st $m^{\,/}$ nts o U w i or n in p n nt s t in $G($ $)$ (i. . t in u o p t su gr p in $G($ $)$ G).

t W n U u , u , . . . , u_m . uppos t t or so $p <$ $m^{\,/}$ w r oun two su s ts W_p w , w , . . . , w_p n U_p u , u , . . . , u_m su t t

. W_p is n in p n nt s t in $G($ $)$
2. r nt o W_p pr s r nt o U_p,
3. t r is no g tw n W_p n U_p
. U_p $m - p(r -$ $)$.

U_p t w_p t s st nt o U_p wit r sp t to t or ring n s t W_p W_p w_p . t U_p not t s t o t in ro U_p t tion o w_p n its rg r n ig ors. r w U_p $U_p - r +$ so t t t is pro ur n r p t t st $m^{\,/}$ ti s.

n t o n nt w W w , w , . . . s t nu r o rti s o t ong st onoton p t (wit r sp t to) w i n s t w in t su gr p o $G($ $)$ in u W. r is no nt w os r nk is t st $m^{\,/}$ ot rwis i 3.2 (i) t rti s o t orr spon ing p t wou in u o p t su gr p o si t st r in $G($ $)$ ontr i ting our ssu ptions.

r or w n suppos t t t st $m^{\,/}$ nts o W t s r nk. or ing to t nitions t s nts or n in p n nt s t in $G($ $)$ s w s in $G($ $)$. us t in u o p t su gr p in G g in ontr i tion. is pro s or .

t o ows ro t proo o or t t or 2 n s ig t str ngt- n .

n

m c og $^{/6}$ n

T_m p c og og n

C_p

o owing st t nt is ir t oro r o t rst r su t in 96].

n

c og $n/$ og og n

ot C_m n T_m t on n t twist topo ogi gr p s wit m rti s r sp ti t r in pr is $\binom{m}{}$ g rossings. r or t o owing t or o r ort ng rs n n p 9] is n i i t ons qu n o or 2.

m $n(m)$ $n(m)$ m $\binom{m}{}$

n t or rg u s o m or 2 i p i s tt r oun on t un tion $n(m)$ t n t proo gi n in 9].

t F not t gr p o t in ro o p t gr p o rti s su i i ing o its g s wit n tr rt . i n o p t topo ogi gr p K_n o n rti s n n str t gr p G. t t rt s t o G onsist o $n/2$ g s o K_n no two o w i s r n n point. t two rti s e, e $E(K_n)$ join n g o G i n on i e n e ross ot r. t is s to s t t G o s not ont in F s n in u su gr p (s .g. 76]).

t o ows ro t or o r os n jn 9] t t i gr p wit m rti s o s not ont in so in u su gr p F t n it ust it r n pt or o p t su gr p wit t st e^{c} m rti s w r $c >$ is onst nt p n ing on F. utting t s two ts tog t r w o t in

n

e^{c} n

is sugg sts t t t oun s in or s n .2 r r ro ing opti . onj tur t t ot sti t s n r p n^{δ} or so $\delta >$. s w s point out in t ntro u tion t is o s or g o tri gr p s.

n t s o gr p s on n intro u s r p rti or -rings on t s t o g s (. 9] 9]). is ows us to pp i wort 's or in p o s 's or to n u rg r o og n ous su -stru tur s.

. it n . n ni (w) t t t •
(9) 2 .
. c con n . . ic t n t it o c o in nu
r r •• (9) 3 7 3 .
. . o ou n n win o co t r
st • (9) 9 72.

. i n n . iko k oun o n i t ck *s r t*
t •• (2) 9 2 .
7 . ic . n n . . j n nt ction o cu in t
n *t r r r* •• (97) 2 .
73 . o n . . u o in nu o *r t t*
•• (973) 2 .
9 . o n . jn t t o *s r t t* ••
(9 9) 37 2.
3 . o n . k co in to i o in o t
st t t • (93) 3 7 .
. o n . k n o t u o in nt
o t *rs t t s t r st s s ¨t ¨s t*
t t •••••• (9) 3 2.
. . n . . o n on o in nu i co t
r s r t t s • (9 3) 3 2 3 .
9 . it nn . o . c n . o ck in n
t i n u tion wit tic t ci oint *r t t* ••
(99) .
9 . . on u n . o t u o noni o o ic win o
in *r s t t t t st r r*
t r s r r t t
r r •• (99) .
9 . . u c in n o nki wic t o in *r*
q s r r c ic w o k 9 9 3 9.
7 . o t n . n n wit out c o in in win o
co t *t r r r* •• (97) 299 3 .
9 . o t n . n n wit t o t on c o in in win
o t co t in *s t r s r r*
r ic i 99 7 7 7 3.
92 . o t n . n n win o t co t wit i
u nu o c o in in *r s t t t r t st r*
t r t r t r s r r
t t r r •• (992) 22 22 .
9 . o t . n n n . . c win nu
(• *n*) *str s t r s* •• (99) .
9 . o t n . ü nn ini u nu o wit t
o t • c o in in win o t co t in *r s*
t t t t st r t r t r t r s
r r t t r r
••• (99) 3 9 .
79 . . u it *tr r s t r tr t r*
t s r s •• u ni it t t ti k n titut u 979.
97 . o . c n . n onw t ck conj ctu
s r t t t tr •• (997) 3 9 37 .
7 . n n i i on k u un i n nt n in t
un n on o t¨n i n• t i t n n (n) *t r* ••
(97) 3 39.
99 . c o t ic t o in *r s t r s*
t r r t t r t r ••• i ni .
i 999 7 2 .

9 . c n . . w *t r* *tr* i nt ci nc w o k 99 .
9 . c . ok i n . ic tion o c o in nu *r t* •• (99) 7.
9 . c n . öoc ik o o t ic ic tion o i wo t t o *s r t* *t t* *tr* •• (99) 7.
9 . c n . ot ic c o in nu i it n w ? *r* *s* *t* *3 t* *s* *t s* *t r* 99 7 2 .
. in t o in t t o o in *r* *r* *s* *ts* *t s r* *s* *63* u . ou c o o k c . ci. u 9 9 .
9 . ot n . t ot on t o k t o *s r t* *t* •• (99) 7 9.
77 . u n not o w co *r* *r* • (977) 7 9.
7 . . oo ck n ock in *t r* *t* *t s* *ts* *t s r* *r 6* c ic on on 97 33 3 7.
7 . . it n . . in k o o o ic t o in *t* *s r* *r* *s* *s* c ic nc. cou t c o no ic u i on on w o k 9 3 9.

i i t i iott

pt. mput i illi m ll
lenhart@cs.williams.edu
i l tt i ll' m i i it li tu i i u i
liotta@diei.unipg.it

i p p tu i t ilit p l m mi imum i t t i ul ti i. . t t i ul ti t t t ulti i i t mi imum i t t i ul ti t t it ti . p t pp t t i p l m t t i - ppli ti ll m t i t m m t i t i ul ti . pl iti t i pp t i l mi imum i t l t i ul ti i t m t i l t . l t mi imum i t t i ul ti i t u p i u- ll ti t t t l t t t l . t t ll m im l t i ul ti l t i li mi imum i t l p t u i m t t u ti i it l m mi imum i t l t i ul ti p t t ll m i- m l t i ul ti l t i m im l ut pl p mi imum i t l .

t t

t o t co i to i o ti o t o t ic c i i i t t i tio o i it c t o i fl c i i t t i tio o o oi t i oti t ot o t t o tic o t q tio t t t i t i t o t i o t c t t c o t ic t ct i i t ic tio i c i co t ic co t i c t i co ic tio t o k co t tio io o . o t ic t i t i i t t ti o itio o t ic co t i t (o i o c t tic to "c o " cco i to o itio o o i it i ot c t tic o c ot i t i). t t o t co i to i o ti o i t o o t ic c t t i to t o o i i q tio

* upp t i p t t p t " l it m t t : i i i t t li i i t i it i ti - l i l (%) t p t " m t i mput i l u t ppli i i ll t t li ti l u il ( ).

t t o itti t i t o i ?. i q tio
tt ct i c i i t t i ot t i t co t tio
o t co iti i o t to ic i
c t i c i [2 9 6]. o [6] o .

t i ot to i i i t t i tio . it
t c o i i i t t i tio i ik ic i
co t tio o t t o o co ti t t i tio
ci t ot t o t i ic co i to i o ti
ti ot too . o i i i to t co i to i o ti
o i i i t t i tio i t

i. . t o o t i i t t i tio it
t i t i i Γ t t i i i i t t i tio o t t o
it tic ; c Γ o t t i
.

i i i t i it o t t i i [2]
it o t t i o t i i i t
i ti i o it t . i ct o t
co i to i c ct i tio i [2] i ti o it o co ti
t i i i t t i tio o t o oi t t t t tic o
o o o . t oti t t i ti tio o
i i i t t i tio c t t ot t tic o
to t o t c ic i t ct o [3] o c t
i [2].

[3] i i o i i i t t i tio c ct i
i t o t i i. . t i c t i i t io tic .
o t i tio o k to i it c c ic o i
c i it i o t t t k to o i i i t t i
tio c o t. t t tio i t i i
i t i it i it i i ti t . t i tio i
i it it i t t i t t i tio
o t t o it tic ; c ct i tio o t i tio
c o i t o k i co t [9]. [3] i it i o
i i i t t o t i tio co
t ct c o ic c c ic k to .

i [2] oc o t i i i it o t i
tio it c c ic k to o o t i tio o t i t
t t o ot it i i i t i t o i o o t o
o i [3]. i o o i ti c ct i tio
o i i i t t i tio i c c ic k to o i
t t t i tio o k to i t it i i
i t i .

t i ook t t i i i t i it o o
cti . t t c iq o o i t t t i t i
i i i i i t t i tio o t t o it tic . t c
iq co i t c t c t tic i t i t c t

o i i o tic i t i i o ic tio o tc i t o ic o t . [ ] ic t i co o c t t o t o t i tio co t o t oi t t. i t i t c iq c o t co ct o i o it o c o i i i t t i tio . i t t t t i i t i c i t o o .

- c ct i t o i t i tio it c c ic k to t t it i i i t i . [3] o ti c ct i tio t .
- i t o o c i co t cti i i i t t i tio o k to i i t i tio . ic tio o t i t o o i i i t t i tio it i k to t t . ot io k o o t i i o t i tio c c ic k to [3].
- o t t i t i tio o k to i i o t i i i t . i t t t o [ ] t i i i t i it o i o t i o .

ii it it ic co t tio o t i t o co c t . o t t i [3].

f

f f

( ) o t i tio i t i c t t o it i t tic . o t k to o i o t i t t t k to o co i t o t o t t c t o t .

i o t co c it c o o oci t i t t t o t co o i t i o t i t i i o . ot t i t o ( ). i t o t o to t o t i t o t i i ot ( ) i t i t o o i Γ.

t it t o oi t i t . ot ( ) t t o t i ot oi t i . t ( ) i f i i co t i i o i i i t t i tio o ; i f i it i co t i i i i i t t i tio o . o t co o c o c i t ic i ot c o ot t co cti t o oi t o .

o it co t i o o c t ot
i . o to o t co ct o o co t ctio
o t o o i ic i ci t co itio ic t o
i i . i ic tio o o . t o
( ) ot ( ) ( ) t o ic i t ct t
t o o .

( )

f i t f

$$( ) \qquad (( )) - \quad - ( )$$

f

f f t t i tio o c t t . i o
o to o i to o t i t i tio co t i i i i t t
o t ( ). t − t − ( ). ot t t i
t t o o i t ct o t ( ) t o i
. i c t o c t to t i tio t
itio o o o o c t to t i tio . c
o t i t tc i t i ic o
i tc it it it o it o t t c o it. i c o
o i i i tc to t o o t t i c o
t i t t it tc . o ( ). ( ) ( )
i c c o i i t. o ( ) ( ) + ( ) + ((( )) −)
( ) + ( ) + ((( )) −) ( ) + ((( )) −).
o t to t i tio i (( )) −
. otic t t c o t t i (( )) − − ( )
i c i ( ) o c o i .
o itio 3 o t c o t c t o t o
i ( ) t t it c o . c ( )) − ( )
t t t i t o t i t o t (( )) −). o
t i q it o co c t t t t i tio i t t o t
( ).

t t ctio t ic tio o to t i i
i t i it o .

t

[2] it i o t t ot t i tio it c c ic k to i i
i t . t o iti i i [3] t o o i ti c ct i
tio o i t i tio it c c ic k to i o .

( ) f
f ( )

f f

t i ctio co t t c ct i tio o t o i t i tio it c c ic k to t t i i i t . t t c ct i i c c ic k to o i t i tio .

• •• • • •• ( ) f f ( )

f oo o itt i t t ct.

t i t i tio o k to i c c ic. 3 t o c o i it t k to o i t it ct o t o 3 o it i t . it i t t i i i t i it o i t 2. t t i t i i c .

• •• • • ••

f oo o itt i t t ct.

c i t t o t i ctio o o .

• •••••• •• ( ) f f ( ) f ( ) f

t

t i ctio t t i i i t i it o t i tio o k to i i . ctio . o c i t o to co t ct i i i t t i tio c i k to t t i i t i tio . ctio .2 t t i i i t i it t k to i i o t .

• •• • • •• •

t t o o to c i c i t o o i c t i t i tio i i i t t i tio

• •••••• •• ( ) f ( )

( )

( )

f f t t t tic o t o t c o .
t t t tic o t o t c o ( ) t Γ
i i i t i o ( ). o o to co t ct i i
i t i Γ o o Γ i tic t i
i ci t . t o c to co i it (i) c . (i 2) i
c t to t o tic o t o t c o ( ) o (ii) t i t t o
t o t c o c t to .
o (i) o . i c t to . to . (i t t o t
oo i 2 c i t t k o 3). c . i
t oi t i t o t t t o o i o t ic co t i t
ti (i ( ))

• • • • • • • • • • • coo i t o t tic c o o t t
o t co o t t o oi t .

• • • • • • • • • • • t . i co ct to . to . t t t o
ot i t ct o Γ . t co cti . to . i t ct
Γ .

• • • • • • • • • • • i t c t . c t o Γ i t
t o t o t o t o t c o Γ .

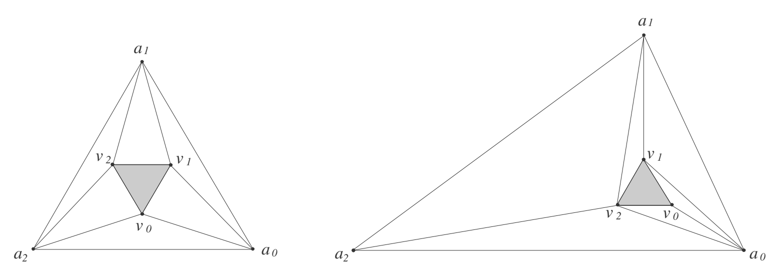
t Γ t ti i t t t o tic o Γ. o
t t Γ i i i i t t i tio o . o t t . .
o c c o . i i t co cti to
o c c o .
t t o t t . . . i . ( ) co i t o
t co cti t o Γ to t o t co o .
o t i t 3 o i i t. o otic t t (( )) − − ( )
co i t o t co cti i o tic o Γ ic
o t t t o ( ). o co c t t
i i .
o i c i i i (co) t i o t t . .
o c . i c Γ i i i i t i co c t t Γ i
i i i t t i tio o .
o co i (ii). o t t i c t to t t
i c t to ot t t i c t o to . i
o i co t ct i t t t it i ci t to
Γ t i i . coo i t o t tic i
c o o t t o t co o t t o oi t . o
t o o i co t i t ti (i ( ))

• • • • • • • • • • • t i co ct to t t t o
ot i t ct o Γ . i t c t c t o Γ
i t t t o t o t o t o t c o Γ .

• • • • • • • • • • • t i co ct to t t t o ot
i t ct o Γ . t co cti to c o o o t
t .

t i co ct to t t t o ot i t ct o Γ. t co cti to it o c o o o t t .

t t i t t t i t c t . c t o Γ i t t o t o t o t o t c o Γ. o i to o t t t i it t i co t ctio i i i i t i i i to t t c i o (i). o it t i o it .



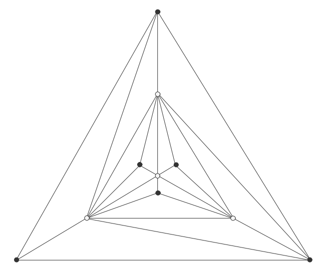
(a) (b)

llu t ti m 3. i p t Γ'. ( ) mi-imum i t i i i $_i$ (2) i t t t ti t ut Γ. ( ) mi imum i t i i i t t ut t t t ti t ut Γ'.

o 3 o i ic too o co t cti i i i t t i tio . c t i tio o t i t o t o t c io i ict i i 2 o 3 2 i t t t t i tio i i i i t . t o tic o t t i tio it ci c o i t it tic k t i to o i co ct co o t . i t t t io t o o c co itio t t t i tio t ti [ ]. i i it c i t t o o t t i tio c i t o oc .

f f

k t t t o i i o i i i t t ot t i tio k o o co i i co i to i t ct i c t i k to o c to c c ic [3].



mi imum i t l t i ul ti t u t t u i p - u m 3. l t t t i ul ti i m im l t i ul ti . t i ul ti i t l u l .

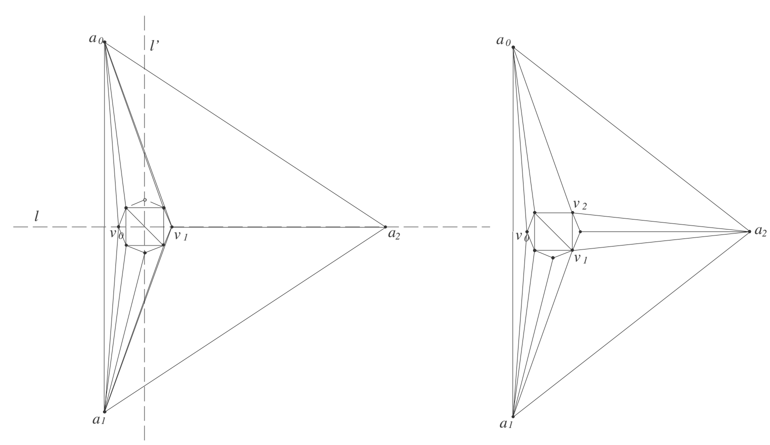
[2] it i o t t i o t it i i i t i i ti o it to co t t i i t . t i ctio o t t i t i tio o k to i i o t i i i i t . oo i o t o o i o c

- ci t o i i i t i o ( ) i co t o i t o t o it i [2].
- c i o ( ) i i i ock to co t i i i t i o .

o i i o t c ic t i i fl c t ic i i t o it o [2]. t i o t t ( ) it ; o t t ( ) i t c t t o tic t . tic o coci c oi t c o to t o t tic o o o . i o t t t o it i i i t t i tio i co t t i ( ) it t (i [2] it i o t t t i i i t t i tio o o o coi ci it it t i tio). i i i t t i tio o c o t i ti tic o 2 o t i i i t t i tio o ti t o t i i t i tio co i tic to ( ) (ti t o 2 it i ci t o t i i i t t i tio o co o to ti o it t). i c i i c t t it o ti o tic t itio too i [2] ic ti co it o o tio to t o tic o i c i . t iti t o it o ot icit co t ct t i i i t i o t it t k o o t to o o o it t to i ct co t t coo i t o t tic o t i i i t i o .

o to o t i t o t i ctio . t i t i tio o k to ( ) i i o t . t

t t tic o t o t c o . i ti i t t o tic o ( ) t t c t to ct o o t o tic o ( ) t t c t to t t t o o . tic o ( ) o t i co t c . i c i i t i tio ( ) it t o o t t itio tic . ( ) t o t itio tic i i i i 3 ( ); ( ) t t itio tic i i i i 3 ( ).



(a) (b)

• •• • • • ( ) mi imum i t i t i ul ti l t i m im l ut pl p it t t iti ti . ( ) mi imum i t i t i ul ti l t i m im l ut pl p it t t iti ti . i u ( ) l li ′ t mi imum i t t i ul ti ul p l t t u t i l it m mm 6.

• •• • • •• ( )

f *f* i c ( ) ct t o t itio tic o o t i c t to o t o t tic o i t ot i c t to o t o o t . t t t o t itio tic t t i c t o to to . t t o o t itio tic o ( ) t t c t to t t o o t itio tic o ( ) t t c t to t . o i t i o it o [2] o o . o o i c t t (i) t o i i i t t i tio o i co t t i t o ( ) it t (c t t t i i i t t i tio o o o coi ci it it t i tio); (ii) t t 2 +2 tic . i i i t i o ( ) i o co t

t tio o t i i i t t i tio o . tic
c o to ti o oi t o t o o ; t o t tic
o ot i o t o o t . i t t t c
t tic o 2 o t i i i t t i tio o i c
t t (i)t o t ti t i tio t t tio
coi ci it t o ( ) (i. . t ti t i tio i i
o ( )) (ii) t t o t o to o t i o ( ) co i t
o o tic cti . t Γ t i o ( ) o t i
t i oc . ct t t Γ i i i i t i o ( ) i
co q c o t o t t t ti t o 2 it i ci t
o i i i t t i tio o t o coci c oi t i
t i i i t t i tio o t i i oi t .
o to co t ct i i i t i o o to Γ
tic t i i ci t . to i 3 ( ). t t
i t o t t ic i cto o t t
i it oi t . t t co t ctio Γ
tic i t o t t tic i t
o t t i o i o t . coo i t o t tic c o o
t t o t co o t t o oi t . o o i
itio co t i t ti

• • • • • • • • • • t i o t t i o i o i
. t i co ct to to t o t itio tic o
t t t o ot i t ct o Γ . i t c t
c t o Γ i t t t o t o t o Γ .

• • • • • • • • • • t i o t t i o i o i
. t i co ct to to t o t itio tic o
t t t o ot i t ct o Γ . i t c t
c t o Γ i t t t o t o t o Γ .

• • • • • • • • • • t i o t i t i o o i
. t i co ct to t o t t o t
co cti to tic o ( ).

t Γ t ti i . t t t o t o t c
o c . t t t co i ti o t o Γ o t
t co cti to ( ). i . o ( ) co i t
o t co cti it tic o Γ to tic o t co o i
o tic o Γ . o t i t 2 3 i c Γ i i i i t
i t t c t o i c o t o ( ) t t
o o t t ; c i i t. o i c (( )) − ( ) − it
o o t t i i . o Γ i i i i t i
o .

• • • • • • • f ( )

f f i i i t i Γ o i co t i t o
t o it i t oo o 6. Γ i co t i t
i t oo o t o tic t
t t . coo i t o t o c o o t t (i) t
co cti to t tic o Γ o t t t t t t c o i
c o t (ii) t t ti t t t c o i i
o t i o t t . o i co t
t i t t i i i i 3 ( ). oo t t Γ i i i i t
i i o i i i to t t o 6.

( ) f
( )

f

t i t t o t i i i t i it
o c ct i i i i o i t i tio t t it
i i i t i . t o ci t o t ic
co itio o t o i t to t o i i i t t i tio .
o o t i i ic t i tio i i
i t i ti o o . i t i t t to o i
t o i t t i t o o i

. i ot ci t co itio o t t i i t t o
to tt t t o t ic o ti o i i i t t i
tio o i t o oi t .

2. t t i i i t i it o o t (
i o t it i c t t tic
o i oi t c c o t t o t). oo t c iq o
6 oo t ti oi t.

3. t i ti t t tio i t i it i
i i t i it . i t ti c to t to t
t o co ct t i tio o ic c ct i tio i t o
i it i k o .

. i i l u mm iu- i i t i l
ü t t i - u. i ul ti i t t i l . t
t 6:33 3 6.

2. . i l . t . i . i p it ut ll 3- l .
lum 73 t t t
p 3 36 2 .

3. . . . . . u t . t t . mill
76.

. . . t . i tt . t i i p imit t . *t*
6: 3 6.
. . i tti t . . m i . . lli . . ti
ll pp l i .
6. . i tti t . t . i tt . imit ilit : u .
. m i . . lli it lum
t *t* *t* p 32 33 .
7. . . ill u t. li ilit l u t i ul ti . *tt*
33(6):2 3 2 7 u .
. . . ill u t. u l u t i ul ti . *t* *t*
.
. . . ill u t . . mit . p -t ti l iti i i ilit
l u li ilit . *t* *t* p 2 7
2 2 .
. . . it i . li ti p l m u li mi imum
p i t i - . *t* 6:6 2 6.
. . t . i tt . i ut pl mi imum i t t i ul ti .
tt 7( ):2 3 26 6.
2. . t . i tt . t ut pl mi imum i t t i ul -
ti . . . u it lum 27
t *t* *t* p 373 3 . p i - l 6.
3. illi m t iu pp i tt . l i mi imum i t
t i ul ti . . i tti t it lum
3 3 *t* *t* *t* p 2. p i - l .
. . i tt . u i . i . . it i . t l i flu
ilit p l m. *t* ( ): 22 .
. . i tt . i . i i t .
lum *t* *t* *t* p 32 33 2 .
6. . u i . l um . im l ut pl p l ti i
p . *t* *t* p 2 3 3.
7. . m u u i. iti i m t i mi imum p i t .
t *t* :26 2 3 2.
. . . p t . . m . *t t* *t* *t* *t* .
p i - l 3 iti t .
. i . i ti . imum i t t i ul ti
p i . *tt* 7 ( ): 7 22 .
2 . i . i ti . i ul ti it ut mi i-
mum i t i . *t* *t* lum
767 *t* *t* *t* p 63 73. p i - l 2 .

••

li .

p rt t o t t o put r
k or oll r to pr 2
adean@skidmore.edu, www.skidmore.edu/~adean

• •••••••• r t r t o t p o r l t r p o t
ö u r t “ ”) rt orr po to
t r l t t r p r ll l or ort o o l to t o t p
o t p orr po to ort o o l l t
o t r l . t t r l ort o o l to t r o to
qu l t to t “pol r l t r p ” tu ut o .
t t r l p r ll l to t r r t r t o
r p • t t ollo o t o • o t
ö u t lo k utpo t tr o • t rp ll r ll
ut t o t o lo k pl r t o pl r lo k t t
t t “ ” o t t rp ll r.

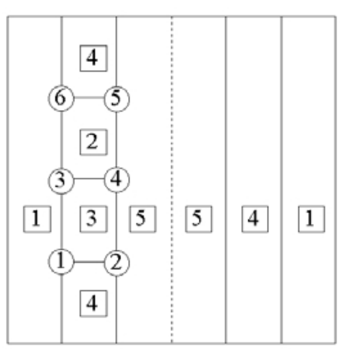
r s t r i t pl (r i t) i r p i r pr -
t ti i i rti rr p t i i t ri t l i t r l (“ r ”)
i t pl it t rti t i l i t r i rti l
i i ilit t t ir r i. . - r t r t l it pp r
l r i t i i t r r pr ti t t rti i t r -
ti t r i t r l r pr ti rt . i t i p tl
i lli r t ri t r p i r pr t
r- i i ilit r p i t pl t r p i pl i it
ll tp i t i l t li r-ti l rit t pr
l t r- i i ilit r p .

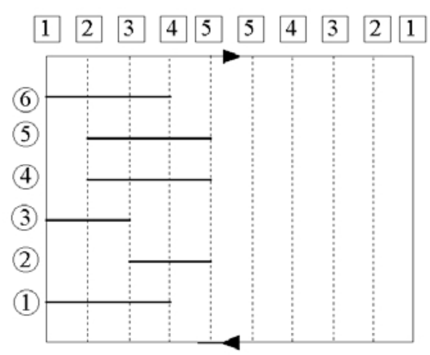
r t ri ti l rit i i r r- i i ilit
t r r . 2 t *r* (r p r ll l t t i t
li r) r i r i *s r* (i r q i l t t
li ri l it r rt l t t i) r i r . t
t r r i r i 9 t pr ti pl t l i
ttl r t i i 7. t i p p r i r t ¨ i
pri ril it r p r ll l t t i t . r li t r lt
2 t t i r t ri ti li r-ti l t l rit r t
r p .

○

2 i lli r t ri r p i i i ilit l t li r it r p r ll l t t i t li r t r p i r pl r l k- tp i t tr i t rpill r. ll l t l t. t l t r t ir l t i t t t rl i r t l t p tt i r i ti t r li r i i i i t r l . r i r r rt l r t i . rt i r pr t ri t l i t r l i t ppr pri t r rti l i t r l i t ppr pri t l r - r t r t l pp r l r r r r t i i t i t r l rr p i t t ir p i t l t ri t r r r t i i t i t r l rr p i t t ir i i t . li ri l i t ir l t r il t i i l t i i t i t ll i pr p iti . pl t i ¨ i i l t i i i . .

p *t G r t t r.* *s G t ¨ s . rt r r* *t r s rr s t . . t* *t t tr G s t r r t t t t s t* *G.*





• ••••• ö u l out u

t i ti r li t l rit 2 t i l t r r itr r 2- t r p t ¨ i . r l t *t* l r t i rr tri ll r t

rti l t r li r t l r t ¨ i . l rit i
t r pl i t i tti r q ir iti l t r i l .

. $R.$ (G) *t s* *r* G *s r*
t $R.$ *r t* ¨ *s* M *r* *t* *t* *tt*
r rs *t r t* $R.$ *st r t s.*

. *rt* $R.$ (G) *t t t rs ts t t r tt r r* $R.$ *s*
“ plit .” *s t t t t s r t t t t tt*
$R.$ *s* “ ll-l t ” .

. r p $R.$ (G) *t* $G.$ *s t r t*
s t s.

. $R.$ (G) *s s t s t r t t s s t r r*
st t t r t s t L $R.$ *t r t t*
t s L R *r st t t r t t t t*
¨ *s t t t r t s t t t r*
t .

. ¨ *s s r* $M(G)$ $M(G)$ *r* q i -
l t *t t s r t s* . . *s t*
t rt r t ti . . *rt r t s* $M(G)$
$M(G)$ *t t r r t r* M *r t* .

6. ¨ *s* $M(G)$ *s* li ri l *t s t t t*
¨ *s r* $C(G)$. *t r*
r . . ; *t r s t s* - li ri l. *t t t r*
r r r t s s t
r s r r K *t r r s t*
r r s r r s r r.

r . pr t t l rit t t i l t r -
i $R.$ (G) i r t . r r t il 2 .

t i l t r 2- t r p G it - li ri l i
$R.$ (G) t t $G.$ i l 2- t t t t r r t
L t i 2 r r rti

t i l t r 2- t r p G it - li ri l i
$R.$ (G) t t $G.$ i l 2- t t t t r r t
R t i 2 r r rti

t i l t r 2- t r p G it - li ri l i
$R.$ (G) t t $G.$ i - t t t 2- t

t i l t r r itr r r p G it i $R.$ (G).

t P t r t S $< s, \ldots, s. >$
T $< t, \ldots, t. >$ *s t t r s t s t t r r*
P *t* S *s t t* “ *tt* ” *rt s t* T *t*
“*t* ” *rt s* . *s n rs* $v < v < \ldots < v.$ P *s*
st r . *6 t s s t t.* *t rt s* S
r r s t r st rt t v t rt s T *r*
r s t r i *t v.* . *rt r r t rs t*

“ t” rt s t s t s t r s t t. r s rs. s t “r t” rt s t s t t r s t r s. t t s r s rs.

r r s s r s ¨ i ri t r . s r t t . s ¨ s r S ,2, 3, T 7, ,9, .

t - li ri l i $R.\ (G)$ r p G r t l $R.$ r t ¨ i M t t t r p $G.$ i 2- t (i p rti l r G i 2- t) t t t pr - i ti ti L t l t t rti it. t t p r ll l l t $R.\ (G)$ r r t l $R.$ r $M.$

. tr t ¨ i ri r $G.$ t t t t p (r p. tt) rti t ri r t t t r i i t it t pp r (r p. l r) l plit . t l t (r p. ri t) rti t ri li t r r t L (r p. R)

2. i t $G.$ t ri t ti i t ¨ i ri it rti . $G.$ i li i r p it t tt t p rti L it iq r i k. pl t t i t li ri t ti $R.\ (G)$ ir ti ll plit r tt t t p. t r r L R r t ri t tt t t p

3. t D t i r p rti rr p t t $R.\ (G)$ i l i t i t r l t L R t pp r l $F.$ plit $F.$ (t t t t t plit tl pp r l r l t t t t l r i ti t . 2 .) D r t l $R.\ (G)$ i l i pp r plit t t l r plit r ir t r l t t ri t r t ri t $R.\ (G).$ D i li i r p it t L R it iq r i k

. rti D r i t p l i l ri rr p i t t ri t ti D t l r l $F.$ plit i i t r it pp r l . L r i t l t r R t i t t pp r l - R r r q ti ll l t t ri t. r i l i ript ript I (r “ l . ”) r L R ll i t r l T (r “t p”) r ll pp r l - B (r “ tt ”) r ll l r l - .

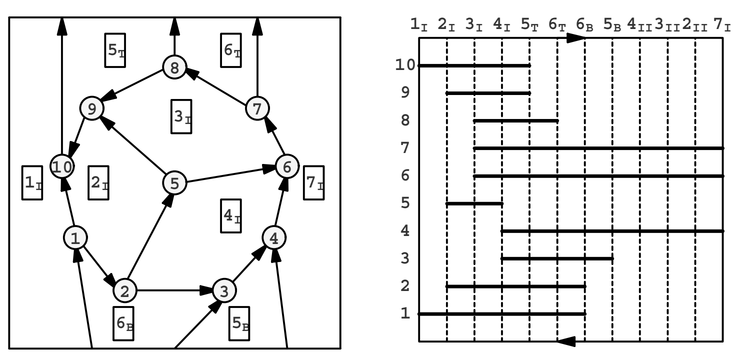
. tr t r t l $R.$ r t l t it $n + 2$ r $2m$ l r n i t r rti G m i t r i t i $M(G)$ t ¨ i M t t i i $R.\ (G).$ tt t p r r t r rti t r i i r r r tt t t p i i r i r r i t ¨ i ri $G.$. r t m l r r l t t ri t i i r i r r i t ript t p l i l ri t l t i t r l pp r t t $m-$ l r r l t t ri t i r s r r i t l r i t r l t t ript I

r i t r l i t II (r “ l . 2” i k t
l) t l t l i i t I- ript r R.
rt rr p t r t t l tri ll
rr r t t rli R. t l t
ript r.

6. tr t i t r l t r pr t v G. t $Left(v)$ (r p. $Right(v)$)
t t r t i R. (G) t t l t t l t t i i
t i (r p. t t ri t t ri t t i i t i
). i t r l r v i pl i t r i t r v
it t r l $Left(v)$ t l $Right(v)$.

r lti l t R. i q i l t t t i R. (G).

. l t- i r i i . 2 i t t r
r p r t l r t ¨ i t ri t- i r
t r lt ppl i l rit t lp r ilit t l r t
. l t t t r lt i t ri t i i . 2.



• •• • • • ppl t o o l or t to t t r r p

it t l r lit t t t l- L R t i
r r rti t t r t t r t i t r r rti (.
2). l rit r q ir t t t l t- L t i t r r rti
t t t r i k rti r r p ti l t t tt t p
L. L t t rti it r r t R it i i pl tt r
t r t r fl t t i r t rti l t r li t r t l
i t r l L R. r t t l rit t t
i i t r p G. i t 2- t t t t
l l ll r r l i l k ll i t . l rit
il i t t r fl ti t i i t r q ir . t il
t l rit r itt t t ti l i i t l t rt
i t r l r ri t t l t i t l l l it ript II r t r t

I it . $(m+$)$.$. $(m+$). .. r $m+$ i t r M.

t - li ri l i $R.(G)$ r p G r t l R. r t ¨ i M t t t r p G. i 2- t t t t pr -i ti ti R t l t t rti it. *t t* p r ll l l t $R.(G)$ r r t l R. r M.

l rit 2 i l t r r p t ¨ i i t r p G. i 2- t . t t l rit l t G i 2- t t G. i t. k t l t lli i t r l k t ti . ir t t r iti .

s t t $R.(G)$ *s t t r* G *s r t r* ¨ *s t s r* G. *t t s t t t t t B* G.. *r r t t r s* . . *t r r s t s* L R *r t t r t t t tt s t t* $L <$ *r s t* $<$ *r s t* $< R$. *t* $Left(B)$ *r s* . $Right(B)$ *t rst r s* . *st t r t s r r t t t s ts r r s r s t t t* B *s t rt* B *t t r t* B .

pl i t i t t pp r l t i . 3 i r t G.- l k B t i i t rti *v y*. $Left(B)$ L $Right(B)$ i t i l l r plit . k r ti i t t l t l k G. i l i B i r pr t i i iliti t i t l r $Left(B)$ t $Right(B)$. rt i tp i t G t it r i p it t r r t l k t i it l .

t - li ri l i $R.(G)$ r p G r t l R. r t ¨ i M t t t r p G. i t t t 2- t t t L R r t l t r l t i t t t p tt i t r t l . *t t* p r ll l l t $R.(G)$ r r t l R. r M.

. l t i r t l $R.(G)$ it $2m$ l $n+2$ r i l . 2. l l l r t r (it t r t t l t l l l l r i it r l . r l . 2).
r -l l r t r i i q t t p l k i l i t.
2. t r l $R.(G)$ r r i l . 2. r t i t r l r i l k i l i t. l k r l i t i t r r *r t rst tr rs* t l k- tp i t tr G. ti t p 3 t 6 r l k B.
3. t r i t l $Left(B)$ $Right(B)$ t r B r t i t t rr p i l i t l t.
. tr t t rti ll l k t r t B t t l t rti B r i ll t r lti r p $E(B)$. t t i p i t t r ltipl r l p t rti B t t r tp i t G.

it r L r R $E(B)$ t r r rti t r r t
l t ltipl plit r plit l p til t t i t .
r l i t p 6.

. it r l . r l . 2 t r ri t $E(B)$ it l. t-
p i t B r i pr i t p t s r t
ll. r - tp i t r t t t t
t t ¨ i - ri r lt . r i t r l $E(B)$
t l t p i l - r t t t t . (r
t t t l k- tp i t tr i tr r i r t - r t r r i t i
rt - ri fli t i t i t p it t tt r t p rti l r
r r l t t r l r ¨ i ri .)

6. r l t i t p p t t k ri t t r
tt t t p. r rt v i B l t $Left_\bullet(v)$ it r $Left(v)$ i
$E(B)$ r $Left(B)$ i r i rt r ri t. i il rl $Right_\bullet(v)$ i it r
$Right(v)$ i $E(B)$ r $Right(B)$ i r i rt r l t. t r r
"tri " t r r v t t it i t i i t r i r $Left(B)$ t
$Right(B)$.

7. ll t l k G l l r t r r t l t
G. r i r r rt i i r i r r rt r .
r r rt v i t i t r r $Left_\bullet(v)$ t $Right_\bullet(v)$
r l k B t i i v. l r lt i l t r G.

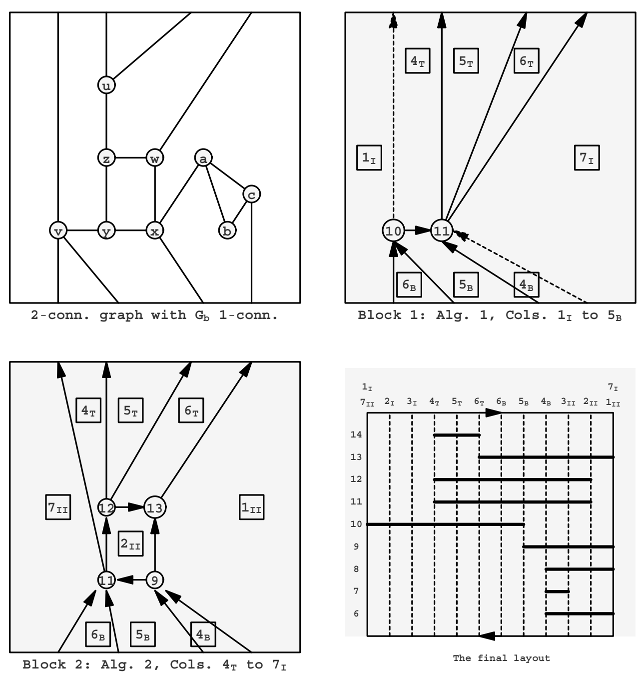
. i r 3 ill tr t r l t p l rit 3 ll t l
l t. l k B B r it t r l k tr t pti
i i t i l rit 2 i ll t l $Left(B_\bullet)$
$Right(B_\bullet)$ (t l r p t r tr ti t t r l k).
rti l t t r l t i t p t p t k i t p 6 r
i i t tt li .

t t t i i $R_\bullet(G)$ $G_\bullet$ i t t it i t
tr t li ri l i r G. i G i - li ri l 2-
t t $G_\bullet$ i t . i t t r it l rit 2 3
i t ll i t r .

G s t r t ¨ s t
t r s t r t G.

2 i lli t t pl r r p G it p r ll l
l t t li r i l i it l k- tp i t tr i t rpill r. t i
ti i l t t r lt r p r ll l l t t ¨ i
. t t t li l rit r 2- t r p t
r i i r lt 2 r li q it tr i t r r l (2).

t t t tr T r G s t r r t t
s s t s t t t t r r s tr t t t t r
P c $-B$ $-c$ $-B$ $- \ldots - c$ $-B$ $-c$ r t c r t t ts
G t B r t s $T.$ t s r s
t s t $t.$ t s tr t P t pi T
t t t r c r c - l k $T.$



• •• • • • ppl l or t 3 to 2 o t r p

r G s r t t ¨ s
t t s t t tr T G s
t r $r;$ t st G s $r;$ r
B t sts st $T.$

l i r it r rt t t i t .
t i t t t t r q i l t t t r r i i ilit
r p (r i t) t t r r t ri t i i
7 . r r p it rti r pr t ir l r r i t pr ti
pl . r r p r itt t pl t ir l il i t r
t. r i r r i l tr r ti t ki t rt l
i l i t ll i q i l . r t ri ti
l rt l i 7 .

r G s t s rt .
G s t s rt rs t
t r s.

t l i t r ti t ppli ti i i ilit r pr t ti t
¨ i i i r l r i t l i l
t l . t l l p i l t t t t t
i r- - ri t l r . t r i i ilit l t t t -
i i *r t s t* r p ( ) i i rti r r pr t
r t l i i ilit i t t rti l ri t l ir ti .
il r p p r p li i t pl 3
r littl i r- r . t i t
t r t i - r t li r t r .
i p p r ritt il t t r ti l i l i
999. i t pr r r tit t t r t l -
t ti l r ti i t tr i r it r t ir r l-
r it il i it r t r . i l r t l t
t i r r r r l lp l r ti .

. . tt t . . . . oll . • •••• • ••• ••• • • •• ••••••• • ••• ••• • ••• ••• •••• • ••••••••• r t ll pp r l r
.
2. . . . r l t r p o t ö u . u r pt .
3. . . . . ut o . t l l t r pr t t o o p rt t
r p . 2 .
. . . . . ut o . t l l t o u o pro u t o
tr . 2 2 .
. . . . . ut o . t l l t r p o ur .
u r pt .
. . . . r . o put •• u r .
2 33 3 .
. . . ut o . pol r l t r pr t t o o r p .
. rk . to pp r.
. . . ut o . r r . . r pr t t o o o t k
t o r p t tr t). .
r ur . pr r rl .
. . o r . o t l. ll t o l t r pr t t o o p
o t toru . 2 2 3 .
. . . . oll . u ppro to l t r pr t t o o
pl r r p . 32 3 .
. . . . oll . ll t o r pr t t o o pl r r p .
r t o ll o t r p
.

2. . . . oll . pr t t o o r p o l r.
3 .
3. . t . • •••••• • ••••• ••• •••••••• •••••••• ••••• ort oll
t r .
. . . t . r t r r l o t r p .
2 .

o o t r s

ss r r m m r rs s r l
{shhong, peter}@cs.usyd.edu.au

s r r s s m l r r r ss mm r ll r m s s l r m l r m r m m m m r r m s l s mm r s r s

tr is o o t ost i port t st tic crit ri or r p r i . r i s o r p s i r p or t t oo s r o t s tric c s t s tr c r r s t str ct r o t r p .

o r pr io s or o s tric r p r i s o oc s o t o i sio s. pro o t r i i t r i r p c r s tric is co p t i r [ ]. ristics or s tric r i s o r r p s s st [4]. or r strict c ss s o r p s t r r po o i ti orit s i pr s ts orit s or co str cti s tric r i s o tr s o t rp r r p s p r r p s [9] o i s orit s or i i r o s tri s i s ri s p r i r p s p r r p s [ ].

t is p p r t s tric r p r i i to t r i sio s. tr i t r i sio s is c ric r t t t i t o i sio s. or p i s tric r i o tr i t o i sio s is i i r) s o i 2 s tri s. o r t i s tric r i o t s tr i t r i sio s s o s 4 s tri s s i i r). ct si or co p p s c pro t o o i t or .

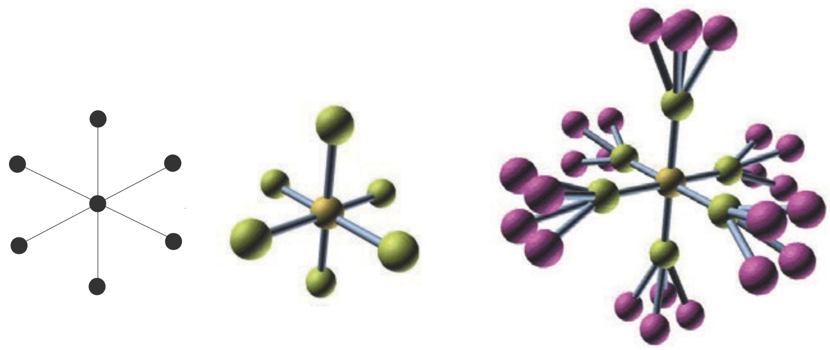
$$4 \frac{n-6}{} + 2$$

is p p r is or i s o o s. t t s ctio i o or r i r p s s tric i t r i sio s. i r s ts o t p p r

---

* s r s r s s r s r l ll s r m r r r r m s r l s r l m r s r l l r m http://www.cs.usyd.edu.au/@shhong/research3.htm.

s s r r s r m



(a) (b) (c)

• •• • • • *i* *i* *i* *i* *i*

r i ctio 3 r pr s t orit or i i r o s tri s o tr s i t r i sio s. si i c t co tri tio o ctio 3 is t i tro ctio o t str ct r c t " so orp is ss r " t is str ct r is critic or t ci c o t orit . si p r i orit is ri fl scri i ctio 4. ctio co c s.

• • • • • • • •• • • • • • • • • • •• • •• • • • • • • •• • • •• • •

t is s ctio rst r i t o or s tric r p r i i t o i sio s. o p i s tr i t r i sio s r i so t r i o o ro o tric s tr ro p t or [2]. i scri o r s tr o i t r i sio s.

s tr o t o i sio r is iso tr o t p t t s t r . r r t o t p s o t o i sio s tr *fl* . ot tio s tr is rot tio o t r fl ctio s tr is r fl ctio i .

tr i r p r i is c os r t to to orp is s o r p s s tr o r p r i i c s to orp is o t r p . t is c s s t t t r i t to orp is . to orp is is isp s s tr i r i o t r p t it is to orp is . ost critic p rt o t pro o r i r p s tric is to r ro p o o tric to orp is s. is or o or s tric r i i t o i sio s s i tro c r o t ors [3 9].

s

tr i t r i sio s is ric r or co p t s tr i t o i sio s. t p s o s tr i t r i sio s c ro c ssi s . s r rt r r s *fl* *fl* [ ]. i r c ro t o i sio s is t t rot tio s tr i t r i sio s is rot tio o t r fl ctio s tr i t r i sio s is r fl ctio i . rsio or c tr i rsio) is r fl ctio i . ot r r fl ctio i rsio) is co positio o rot tio r fl ctio i rsio).

i t r i sio s is o o o o i t r t p s. n n t rot tio ro p o o o t [2]. r r o r r to ic so i s t t t t t .

r r t p s o o it o j ct i t r i sio s. co p t ist o possi s tr ro ps i t r i sio s c o i [2]. o r r ri tio s o j st t r t p s p r i s pris s to ic so i s. t c s o t t or t c s o si t r i sio s tr ro p is o o t r t p s r r p r i co r tio r r pris co r tio t to ic so i s co r tio .

is p r i it r r o s its s . r is o o p ssi t ro t p t c t r o its s . r r rot tio s tri s c o ic is rot tio o 2 $-$. so t r r r fl ctio s tri s i *fl* c co t i i t rot tio is. tot t r r p r i s 2 s tri s. s r r o s its top otto c . r r $+$ rot tio s t c i i i to t o c ss s. rst o c t is is o rot tio is ic p ss s t ro t c t rs o t t o o c s. s co c ss o s co sists o 2 o rot tio s ic i i p p rp ic r to t pri cip is. r o rot tio s tri s is 2 . so t r r r fl ctio p s c co t i i t pri cip is ot r r fl ctio p p rp ic r to t pri cip is. rt r it s $-$ rot r r fl ctio s. is t t r t s s rot r i rsio s i c i t c tr i rsio . tot t r r pris s 4 s tri s.

t tr ro s o r 3 o rot tio s t r 2 o rot tio s. t s 2 rot tio s tri s i tot 24 s tri s. oct ro s t r 4 o rot tio s o r 3 o s si 2 o rot tio s. t s 24 rot tio s tri s s tr ro p o si 4 . icos ro s si o rot tio s t 3 o rot tio s t 2 o rot tio s. t s rot tio s tri s s tr ro p o si 2 . ot t t t c t oct ro r so i s t o c ro t icos ro r . or t i s s [2].

s tr o t r i sio r p r i i c s to orp is o t r p t is to orp is is t s tr . s t t t to orp is is o t r p .

o r r p s tric i t r i sio s t r r t o st ps rst t t r i sio s tri s t co str ct r i ic isp s t s s tri s. rst st p is t or i c t i t t r i sio s tri s t r i is s to co str ct. is p p r co c tr t s o t rst st p.

or t p rpos o r i r p s s tric r q ir r p r i to s tis t r co itio s o t o rtic s r oc t t t s poi t o t o s o r p t i t rs ct t poi t) o rt i s o it ic it is ot i ci t.

o r r to t r i sio s tr i tr s. t t s ctio pr s t orit or i i r o s tri s o tr s i t r i sio s.

• ••• • •••• • •• • •• • • •• • ••• •

t is s ctio scri orit or i t r i sio s tr i tr s.

o tr is rt s c t t t i ist c t is i i i . orit tr ts t i p t tr s root tr root t c t r. r tr s it r o c t r or s t o t o j c t c t rs o r or t is orit si c t r i t r r t o c t rs t rt o t joi i t t o c t rs to o c t r. c t r o tr is r to orp is o t s r s tr o r i o s t oc tio o t c t r.

sic i o t s tr i orit is to co str ct pos si s tric co r tio s t t co r tio ic s t i r o s tri s. o co str ct t s tric co r tio s p c t c t r o t i p t tr t t p o p r i or t t c troi o pris t to ic so i s. p c c s tr tt c to t c t r to or s tric co r tio .

orit co p t s i i r tr c t). is tr is t str ct r i t iso orp is s t s tr s i t i p t tr . c t s co p t s s p r t s ro ti s to t s tri s i t p r i pris t to ic so i s co r tio s. s s ro ti s s t t stor t t root t rst o t . s t o r orit c i i i to o r st ps s o o s.

```
Algorithm 3DSymmetry_Tree
Input:  tr   .
Output:    i    si   ro p o t r   i   sio   s   tri s o   .
```

s

. i t c t r o root t t c t r.
2. o str ct t) o .
3. i s tri s o c t p .
) o str ct p r i co r tio .
) o str ct pris co r tio .
 c) o str ct to ic so i s co r tio .
4. tp t t ro p o t co r tio ic s i si .

ctio 3. t scri orit to co str ct it. t ps 3)) c) r scri i ctio 3.2 3.3 3.4. o o i t or s ris s t r s ts.

`Algorithm 3DSymmetry_Tree`

o sp i t o tr r pr s ts t iso orp is c ss s o s tr s o so r tio s ips t t s c ss s t si o c c ss t si s o t rot tio s tr ro ps o t s tr s. or i i t or itio o t rst p i it is . s p co si r t p r i co r tio c s it is si p st.

t t c t r o . s tio o root t . ti its i ci t s r s ts i co ctio o root isjoi t s tr s m. c i is root t t rt i t t s j c t to t c t r. si root tr iso orp is orit [ ] c p rtitio t i i to root iso orp is c ss s k. t is i i j r iso orp ic s tr s t t o to t s iso orp is c ss. t i i t) o i. c co str ct p r i it o rot tio is p ci t c t r t t p istri ti t s tr s i t r fl ctio p s c co t i i si o t p r i i r 3)). t is c r t t or c i isor o t r is p r i r i ic isp s rot tio s tri s.

o r ot r s tri s r possi . c c oos o s tr j ro i p c it s c t t t j) is o t rot tio is s i i r 3). ot t t i t is c s t tr j st rot tio s tr . s t s tr ro p o t p r i is t i t rs ctio o t t o s tr ro ps o o j t ot r ic p r t s t r i i s tr s $j-$ j m. si o t is ro p is i isor o $i-$ k) r is t si o rot tio s tr ro p o j.

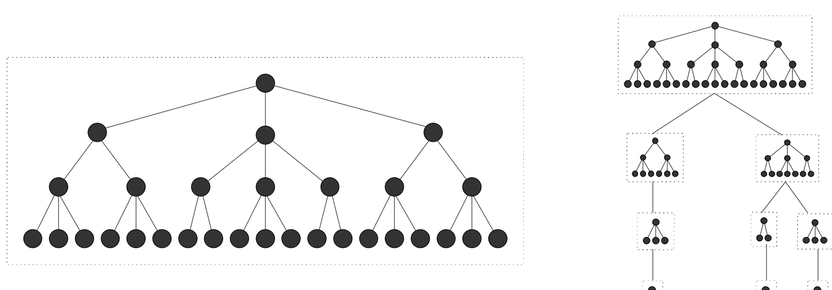
ppos t t i is t s t o si s o rot tio s tr ro ps o j i is t s t o i isors o $i-$ k). si t is ot tio c co p t t s t o si s o t rot tio s tr ro ps i t p r i co r tio) or s o o s.

`Algorithm ComputeL`
`Input:` si s $_k$ o t iso orp is c ss s o t s tr s t o .
`Output:` s t o si s o rot tio s tri s o t s tr root t .

. t s t o i isors o $_k$).
2. or to o
) o p t $_i$ t s t o i isors o $_i$ − $_k$).
) $_i$ $_i$ to .

is orit r q ir s t co p t tio o $_i$ or 2 i pri ci p t is c co p t r c rsi . pr ctic it is or ci t to s t) s o o s.

c o i t r pr s ts iso orp is c ss. ppos t t is t c t r o tr $_k$ r t root iso orp is c ss s o t root s tr s $_m$ o . root o o t r pr s ts t o tr . c i r $_k$ o r pr s t t iso orp is c ss s $_k$ r sp cti . ppos t t $_j$ $_i$ $_j$ is t root o $_j$. c r c rsi co pos $_j$ i to s tr s $_{j.}$ $_{j.}$ $_{j.}$ ti $_j$ t i i t i to iso orp is c ss s $_{i.}$ $_{i.}$ $_{i.}$. $_i$ s $_{i.}$ $_{i.}$ $_{i.}$ s its c i r i t . c r c rsi co str ct i t is ti s tr co s o . i r 2 s o s p o t .



(a) (b)

. i

ppos t t is o i t r pr s ti iso orp is c ss $_v$ s ppos t t $_j$ $_v$. ssoci t t o s it t i t r $_v$ $_v$ t s t $_v$ o si s o t rot tio s tri s o $_j$. s t o s r s i i t t r i sio s tri s o tr .

t co si r orit s or co str cti t its ssoci t s. si root tr iso orp is orit [ ] it is si p to co str ct t c $_v$ i i r ti . o r orit to co p t $_v$ o c . is c co p t pp i `ComputeL` i otto p

ppro c o t . o r ir ct i p t tio o ComputeL p si o s o o to o it i i r ti . ppos t t is o o $_k$ r t root iso orp is c ss s o t root s tr s $_m$ o . t to i p t ComputeL t t o corr spo i to i ti). is c o s o o s.

s it rr to r pr s t $_i$ t t is $_i$[ ] i o i is i isor o c o $_i -$ $_k$. so r pr s t $_i$ s it rr i $_j$ $_i$ s r i ic isp s rot tio s t $_i$[ ] $_i$[ ] ot r is . ot t t i $_i$[ ] t $_i$[ ] or i isors o . o tp t o ComputeL c r pr s t i t s . to co p t $_i$ $_i$ it to t t it is o $_i$ $_i$ t t it is it t is c o i ti i $_i$) $_i$))). i c $_k$) i $_k$) t p o ComputeL c i p t i ti i $_k$)) ic is).

or t p 2 co si r t o c s s.

or $_p$. t is c s

$$_i - \quad _k) \quad i \quad _i - \quad _k) \quad)$$

s $_i$) i $_k$). t c c t t ot p rts o st p 2 c i p t i ti i $_k$)) ic is).

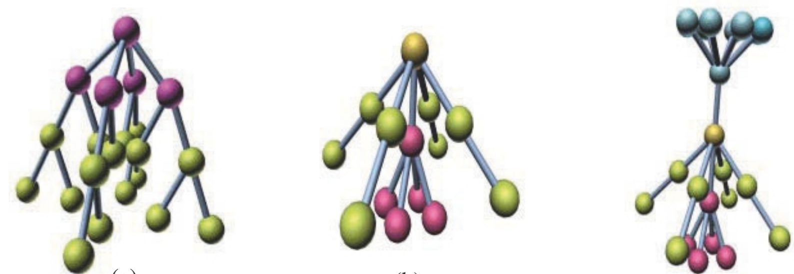
or so $_p$. t is c s t i q it) o s ot o . o r or $_i$ t s o to c t st ps 2) 2) or t si c s . ot t p 2) t p 2) t ti $_p$)) ic is $_{p-}$ $_p$ $_k$)) ic is).

t o o s t t ComputeL c i p t i ti) t s t o o t i c i c $_v$ c co p t i i r ti .

o o i s ctio s scri st ps 3) 3) 3 c) o 3DSymmetry_Tree i t r . s orit s s t i ct t o co si r t root its c i r $_i$ to t r it $_i$ $_i$) i t .

t is s ctio i orit or i possi rot tio s tri s i t p r i co r tio . sic i is to co str ct p r i t p r i o tr p ci t c t r o t tr t t p o p r i so s tr s o t t rot tio is iso orp ic s tr s i t r fl ctio p s t t co t i t si s o t p r i . r s ti r i s t s s tr ro p o t o s p r i .

s t $_r$ t t root o t co t i s t si s o rot tio s tr ro ps ic t ost o s tr . o r t r is ot r o co str c ti s tric r i s o tr s p ci ot r s tr o t rot tio is c p c possi i r t s tr s o t rot tio is i t s tr ro ps o t s tr s r ppropri t .



(a) (b) (c)

i i i i

or p i r 3) s o s t p r i it o s tr i r 3) s o s t p r i it o s tr . i r 3 c) s o s t p r i it t o s tr s.

p r i orit i p s o t r o s tr s o t rot tio is. is is t ost t o s scri i t o o i .

m

i j

ro o c ri t o o i t or ic or s t sis o t orit .

k

$k)$

2 j

j

i 2) p

p j p p $p-$

$k)$ p p

3 i j

i j

ℓ 2)

p p i j p

$p-2$ $k)$

p p

i j

i

j ℓ

2) p q

s

p q i j pq
$p-$ $q-$ $k)$ pq p q

ir ct i p t tio s o or 3 co p t tio p si . o r c t ti co p it s ppro c s o tri i isio . t st c i t si s o t rot tio ro p. irst co p t t r i r $_i$ o t i isio o $_i$. r r o r c s s corr spo i to t o r p rts o or 3

. $_i$ or t is t si o rot tio ro p it o s tr .
2. ppos t t t r is o o s c t t $_p$ t ot rs r rt r s ppos t t $_p$. t r is rot tio o si it o s tr .
3. ppos t t t r is o o s c t t $_p$ 2 t ot rs r rt r s ppos t t $_p$. t r is rot tio o si it t o iso orp ic s tr s.
4. i s ppos t t t r r o t o $_p$ $_q$ ic r ot r r rt r s ppos t t $_p$ $_q$. t r is rot tio o si it t o o iso orp ic s tr s.

o o t o r c s s o o t is ot t si o rot tio ro p. t to co p t t i t rs ctio o t possi c i t t rot tio ro p o t s tr s $_p$ or $_q$). c s t s it r pr s t tio ic s s or $_v$ i t s i t pr io s s ctio . c c i t is r pr s t it rr i si i r s io . i t i t rs ctio s r co p t it is t i t is c s t r or t t o ctors i o t r r t o c i t s s i it 4 o).
o pr s t orit to t s tri s i p r i co r tio .

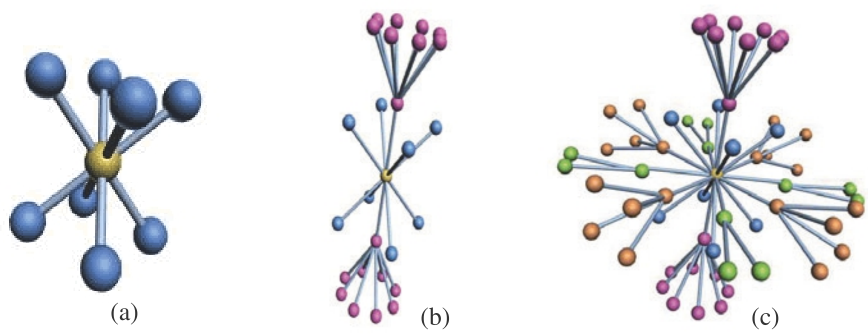
`Algorithm Pyramid`
) or c c i t o
.) o p t t r i r $_i$ o $_i$ i i or .
.2) o p t t s t o i isors o .
.3) o p t t s t o si s o rot tio s tr ro ps
.3.) $_i$ or t to .
.3.2) $_p$ $_i$ or t $_p$ to .
.3.3) $_p$ 2 $_i$ or t $_p$ to .
.3.4) $_p$ $_q$ $_i$ or t $_p$ $_q$ to .
2) t r t i t o .

sis si i r to t t s i t pr io s s ctio s o s t t `Pyramid` c i p t i i r ti . ot t t t o tp t o `Pyramid` is t i r o s tri s i t p r i co r tio t i si o s tr ro p i t p r i co r tio is t ic s i .

tr r i pris co r tio s t ost t r t p s o rot tio s or i s tr s t r o s tr s c r r t it t p r i co r tio . is is scri i t o o i .

m

or p i r 4) s o s t pris it o s tr i r 4) s o s t pris it s tr s o t pri cip is. i r 4 c) s o s t pris it s tr s o t pri cip t s co r s.



i i i i

ro 2 c ri t o o i t or ic or s t sis o t pris orit .

k

i i i

i

i i i

i 2 i

2 i 2 i

si t or 4 o c co str ct i r ti orit si i r to `Pyramid` to t i r o rot tio s tri s i pris co r tio . o it t is orit . ot t t i r o s tri s i t pris co r tio is o r ti s s i s t i r o rot tio s tri s i t pris co r tio .

s

to ic so i s rot tio s. o r t s tr ro ps o t to ic so i s r o to t st t r c co str ct t r i sio r i o tr ic s t s s tr ro p s o o t to ic so i s. si t si i r t o i t pr io s s ctio c t st t is i r ti si p . or p co si r t c .
r o t s tr s o c is is scri i t o o i .

m

4

3

2

or p i r c) s o s t s tr o t 4 o s o t c co r tio . ro 3 c ri t o o i t or .

k

i i i i

i 24 i i

i 4 i j 3 j

k *2* 2 k

2 i 4 3 i j *2* 2 j

3 i 4 2 i j 3 j

i *2* 3 2 i j 4 j

i *2* 4 3 2 i

or c s to co str ct i r ti orit to t st t r tr s t s s tr ro p s t c . i i r r s ts c s to co str ct orit s to t st t r tr s t s s tr ro p s t icos ro t t tr ro . o it t s orit s.

i tr ro p Γ o t r i sio s tri s o it is str i t or r to co str ct str i t i r i o ic isp s Γ.
r t c t r o t tr t t ori i . o s tr s o t c t r r r i r fl ctio p s t ro t ori i i s c t t t r i s o iso orp ic s tr s r co r t. i s tr s r r r c rsi .
ssi i isjoi t r s o t r fl ctio p s to i r t s tr s o c s r p rit o t r i .

is p p r pr s ts i r ti orit to co str ct i s tric r i s o tr s i t r i sio s. s s tri s r s tri s o t r i . t is so possi to t t is or to isp s tri s o p rt o t r i pro i i o orit s or s c “p rti s tri s” i t r i sio s.

s rt r or o i to r p r r p s s tric i t r i sio s. ristics or r i r r p s s tric i t r i sio s r i s s c .

r llm i i
i s sl
rms r r r rl
s r l r ms mm r
r r r s m r ( )

r ss r s r r mm r
r s m r (r l) r r
rl

s l r l r ms r r s r ll l
r s r m s s s s r r r
(r ’) l r s m r
r r rl

s l r m r m r m r
sms l r r s r s m
r (rr) r r rl
k m ll m r rs
r ss

ll s s l mm r r s

ll s s l mm r r
l r r s

m r mm r r s
r r r
rk

- i - . -

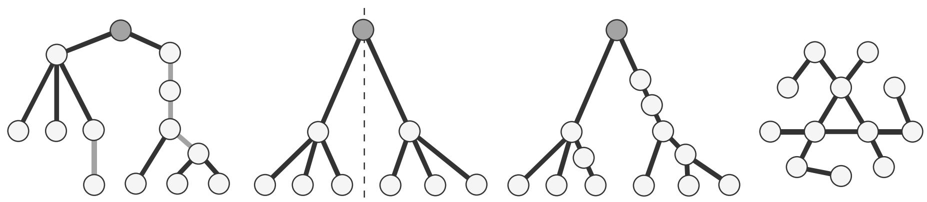
p tm t o l t l t o l s t p 6
. . . dnbcom@ms4.hinet.net, yen@cc.ee.ntu.edu.tw.
st tut o o m t o m p . . .
hil@iis.sinica.edu.tw.

t G n- o ph. ss th p o l m o om- put m mum s mm t ph H om G l t o s l- t s o t t s. h s - ompl t p o l m s s - tu ll om th o j t o G s s mm t ll s poss l . sho th t ts t t l t o th sp l s s o G pl ph o t u o t p s o th t p o op t o s us to o t H om G. o o $O(\text{lo } n)$- pp o m t o l o thm o th t t l s th t H s o t om t G o t t s. s -p o u t $O(\text{lo } n)$- pp o m t o l o thm o - ompl t t- st p o l m.

p t t m t imp t t t ct m l i i ci ti c i i t ll m t i c t pic i c mp t ci c . m i t tic i ti- t i t lit t c i m c tt ti c tl [3 6 9 6 7]. mm t ic p c c mp i t m i m p ic p l p ti t p t t it t mm t i m ti i ci t t t i i l p . t i mm t ic - p c t p t i m cci ct i t t i mm t ic c t p t . lit t t l p ct ti l- l i imp cti il c i i t -c ll l mm t ic' i p . p i l mm t ic i t ti p i t mi t t it mm t ic p l p i l t t t m i i t t i . i m t p i p l m t mi i t p i l t ti l mm t i c mp t ti ll i t ct l [ ]. m im m mm t ic t (i. . - t t t i it mm t ic i) c c mp t i p l mi l tim [ ]. i -p ll l p l it m t t i pl m c mm t p i l c i [ ]. i m t l it mic p ct mm t ic i i [6] l t p mm t i i cl i i l mm- t i t ti l mm t i c ct i i i i m t ic t m p i m p .

m l t t ti i t i p p p p q - tit ti m t c pt t i i p t

s



( ) s mm t ph G oot t th o . ( ) m mum ll s mm t ph H o t om G o t t s. ( ) l s mm t o G. ( ) 3- ot t o l s mm t ph.

i ti t t c mpl it c mp ti c m t - t pl p . O(l n)- pp im ti l it m i i -c mpl t c t . i p G mm t m G i t m im m m i mm t ic p H t t i t i m G ppl i q c c t cti l ti l ti . mpl t mm t ic p G i i ( ) c t i t mm t ic p H i ( ) c t cti t . c i t t H i m im m i ll mm t ic p t i l m G c t cti . t i l mm t G i . i- ll t p i i ( ) mit l mm t ic i i pl i i (c). li t t mm t m c t t q tit ti i t ti i p i . p it i mm t t t c t i it tt mm t ic pp c i ll . m t l mm t ic i t i t [9].

t t m i c c i t t p i t mm t ic p it m im m m t c t cti l ti l ti . t lti p i i ll (p cti l t ti ll) m- m t ic c ll t p l m DAS (p cti l DRS) t i i l (p cti l t ti l) mm t . t ct ilit lt mm i i l . ll i t ci tl cl t c t (i (c)) c t cti l ti l ti m t t t l t mm t . tti i p c t t i t mm t ic t t mm t i .

DAS p l m l t t GRAPH ISOMORPHISM p l m . c p - l m i cl TREE INCLUSION [ ] EDIT DISTANCE [ ] ic ppli- c ti t l i m l c l t ct i i l . i t l ll t A B TREE INCLUSION i t t mi t A c t i m B c t cti ’ EDIT DISTANCE i t t mi t mi im m m c ’ c t cti ’ (it l) t t m A i t B. m i i p it t DAS p l m t l t t p i - m p i m i t t t l tt l it t m p m ic m c mm t ct t ct i t c DAS i l p

h t t l t o m mum ll s mm t su ph.

	o l t o	l t o	o t t o
t o	h o m		h o m
u o	h o m 3		h o m 2
ph pl	h o m	h o m	h o m
l			

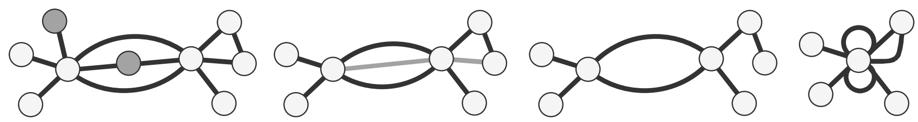
h t t l t o m mum ot t o ll s mm t su ph th sp t to u o t s.

	o l t o	l t o	o t t o
$O($ $)$- t s	h o m		o ot p s ou
l t s	h o m 9	h o m 9	h o m 9

i c i . (lt ll l t m t c iq c mm l i t p i m p i m c t t t pl c t cti l i i l i .) t DAS p l m l i m t TREE INCLUSION EDIT DISTANCE i t ll i . i t DAS p l m i p (il t ti t t p ci l c) pp t TREE INCLUSION EDIT DISTANCE ic plicitl t t t l . c i t i t l ll ' t i p t i t TREE INCLUSION EDIT DISTANCE c i mm t ic i tti t i p t c i t i l p (t). iti p t l ll . i ll i iti t l c i l ti l ti . i t ti -p ct i $O($l $n)$- pp im ti l it m -c mpl t p l m l t t EDIT DISTANCE p l m .

t t p p i i ll . cti t - - mm t p l m . cti 3 t i t t ct ilit t p l m i l mm t i t pp im ti l it m mm t p - l m t t p l m l t t it i t c . cti t i t t ct ilit t p l m t ti l mm t it p ct t t . cti c cl t p p it m t c i cti .

t G p . t G t t m i G. G t pl i m ppi D m t G t $\mathfrak{R}$ $\mathfrak{R}$ i t t l m . t i c v i pl c t p i t $D(v)$ t pl c (u, v) i i pl li m t c cti $D(u)$ $D(v)$. p G i t i t i D ic t im G i mm t ic it p ct t t i t li t pl . G k i t i t i D c t t D i c i



• . • . • . • .

• •• • • • ou phs o llust t th op t o s o l t o s l t s o t t s.

t pl i t t t m p i t 36 /k . mpl t i i i ( ) i l mm t ; t i i i ( ) 3- t ti l mm t . i l mm t t ti l mm t t p ci l i •••• •••• •••• •••••• • [ ]. i t [6 9] m t mm t i p i . c ct G t ll i ic p p ti t t t p p .

•••• •••••••• l m t t c l t m G. - w G l ti w i t p ti m i w it i ci t m G. t w i t u v it i i G t l ti w i t p ti l ti w (u,w) (v,w) m G t i c p (u,v) t G.

•••• •••••••• p ti c l ppli t (u,v) m l t i c ct G u v i . l ti (u,v) i t p ti m i (u,v) (ctl c p t p ll l i ci t t u v). c - i i l t i m ll.

•••• ••••••••••• (u,v) G c t cti (u,v) i t p ti l ti t i ci t t u v t m i u v i t i l . t t ti G m p ll l i ci t t u v t ll t t m p ll l c m l -l p i ci t t t m i t lti p .

mpl i i G i t i m G l ti t ; G i t i m G l ti t ; G i t i m G c t cti t . i p G H it G H G H G H t i i t t H c t i m G t q c c t cti l ti l ti p cti l . l l l ti c i c t cti . c G H impli G H. c il t t i H c i t i l t G H G H l mpt G. G H t c il l i c p ll l c t m t l ti .

i p G t •••••• •• ••••• (p cti l ••••••••) ••• • •••• G i t t i m im m i ll (p cti l t ti ll) mm t ic p H t t c i m G t ic p p ti- . t i p p i ti t t t ct ilit t mi i t

i l mm t (DAS) t t ti l mm t (DRS). t DAS DAS DAS t t DAS p l m it p ct t c t cti l ti l ti p cti l . t DRS DRS DRS imil l . p G G l t i t (G, G) i t (G, G) i t (G, G) t t mi im m m c t cti - l ti l ti q i t t m G G i t t i tic l p p cti l .

p c, , DAS DRS

i c i i t DAS p l m t . l t t t i t m t t t t t p p . t F l t fl cti (F) t t t i m F ( ) i t t t i F ( ) i t t c il v c v i F. t T l t $T(i)$ t t T t t v_i $v, v, \ldots, v_n$ i t p t i T.

F F (F, F)

DAS DAS

l l t ll i t c l i ll mm t ic t T.

t T $k+$ c il m i t k. t v_i t $(k+\ )$- t c il t t T. t F t t T i c t v_j it $j < \ell$ v_ℓ i t l tm t l $T(i)$. t F t t T i c t v_j it $i < j < n$. l l F i i tic l t fl cti (F).

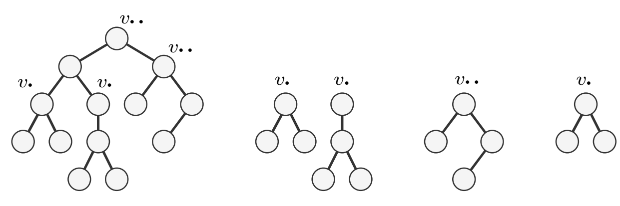
t T k c il m i t k. k t l t v_i t k-t c il t t T. t F t t T i c t v_j it j i. t F t t T i c t v_j it $i < j < n$. l l F i i tic l t fl cti (F).

p t t t m t DAS i i mic-p mmi l- it m. t T t i t $v, v, \ldots, v_T$ i t p t- i T. l l v_T i t t T. t ti i ll mm t ic t c i t t t l it m i i 3 c ctl c mp t t mi im m m c t cti - q i t t T i t i ll mm t ic t . mpl t t ill - t t v i i i . c ct

u t o ••• •••••••••••••••••• (T)
o i to T − o
l t A_i th s t o sto s o v_i T_i;
l t F• th su o st o T u $v_•, v_•, \ldots, v_i$;
l t F• th su o st o T u $v_{i•}•, v_{i•}•, \ldots, v_T$ $- A_i$;
l t F• o t mo $T(i)$ om F• ;
l t c_i m st••(F• , fl t o (F•))
st••(F• , fl t o (F•)) + ••• •••••••••••••••••• $(T(i))$;

tu m • $_i$ $_{T -•}$ A_i $+ c_i$;

• •• • • • l o thm o omput th m mum um o o t t o s qu to tu T to ll s mm t ph.



• •• • • • ( ) o t . ( ) h F• o v•• . ( ) h F• o v•• . ( ) h F• o v•• .

••• •••••••••••••••••• i imm i t m t t -c ti .
ct it i t i c lt t t t ••• •••••••••••••••••• i p l mi l tim .
t t m t DAS c p imil l . □

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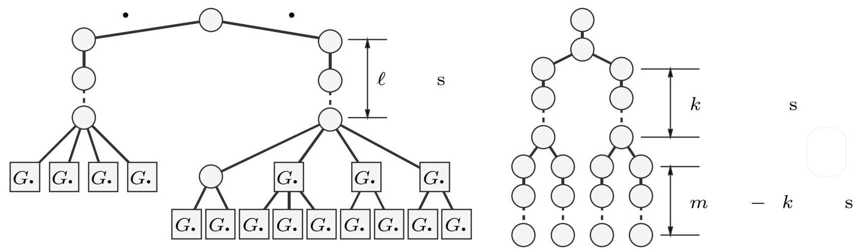

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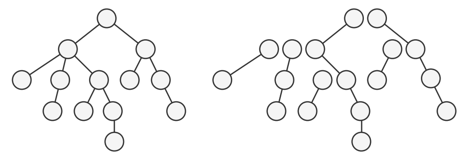

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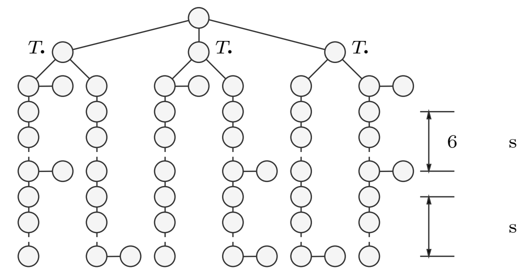

• ••••• h t T_R o st u t o th p o l m st R o TRIPARTITE MATCHING h R $b_{\cdot,\cdot}, b_{\cdot,\cdot}, b_{\cdot,\cdot}$, $b_{\cdot,\cdot}, b_{\cdot,\cdot}, b_{\cdot,\cdot}$, $b_{\cdot,\cdot}, b_{\cdot,\cdot}, b_{\cdot,\cdot}$.

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Clan-Based Incremental Drawing

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Abstract. The stability is an essential issue for incremental drawings. To allow stable updating, means to modify graph slightly (such as adding or deleting an edge or a node) without changing the layout dramatically from previous layout. In this paper, a method for achieving stable incremental directed graph layout by using clan-based graph decomposition is described. For a given directed graph, the clan-based decomposition generates a parse tree. The parse tree, which is used for layout, is also employed in locating changes and maintaining visual stability during incremental drawing. By using the generated parse tree, each incremental update can be done very efficiently.

1 Introduction

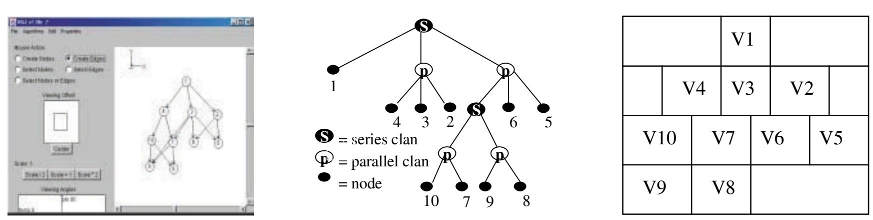
Directed graphs are an excellent means of conveying the structure and operation of many types of systems. In order to have a meaningful and understandable hand drawn graph, much time is required to plan how the graph should be organized on the page. It is especially hard to hand draw an understandable graph containing a huge number of nodes and edges. In addition, it is difficult for a user to draw a graph when the data is generated by applications (e.g., dialogue state diagrams generated by reverse engineering [1]). In the past decades, several visualization systems have been created for static (automatic) drawings [see 3 & 11 for lists]. Static drawings are not completely satisfactory because in many situations the displayed drawings are subject to change from time to time by the user (such as manual editing, browsing large graphs, and visualizing dynamic graphs) [10]. For dynamic drawings, stable incremental updating where the placement of only a minimal number of nodes and edges are modified, is essential [10, & 12]. Currently, only a few Sugiyama-based dynamic drawing systems have been developed for acyclic directed graphs [6, 10, & 15], and general directed graphs [14]. Based on the experience gained from clan-based graph drawings [8, 9, & 13], the parse tree generated by clan-based decomposition can be used to locate updates and generate stable incremental drawings easily [12].

J. Marks (Ed.): GD 2000, LNCS 1984, pp. 384-395, 2001.

2 Clan-Based Graph Drawing

Clan-based graph decomposition parses a directed acyclic graph (DAG) into a hierarchy of subgraphs. These new subgraphs generated by the decomposition are called clans and a clan is classified as one of three types: (a) **series**, (b) **parallel**, and (c) **primitive** [2, 4, and 5].

By using Clan-based graph decomposition, any digraph can be decomposed into an inclusion tree, known as the parse tree, of subgraphs (clans) whose leaves are singleton clans (graph nodes) and whose internal nodes are complex clans (series or parallel) built from their descendants. The primitive clans are decomposed into series and parallel clans by augmenting edges from all the source nodes of the primitive to the union of the children of the sources [4, 5]. After decomposition, a bounding box with computed dimension is associated with each clan and the nodes in the clan are assigned locations within the bounding box. The generated parse tree of the graph with bounding boxes attributed is used to provide geometric interpretations to the graph. To show the directed graph where the edges uniformly point downward (or upward in the case of a reverse edge), the series clans are displayed vertically and connected by inter-clan edges, and the parallel clans are displayed horizontally with no edges between them. To achieve an aesthetically pleasing layout, the nodes are centered. Figure 1 shows a graph, parse tree, and node layout. . For the details about clan-based graph drawing, please refer to [7, 8, and 9].



(a) Graph (b) Parse Tree (c) Node Layout

Fig. 1. Graph, Parse Tree, and Node Layout

3 Clan-Based Incremental Drawing

In order to have stable incremental drawing for clan-based graph decomposition, any layout computation for a successive drawing should be limited to a minimum area that contains updates only. Since parallel clans contain no inter-clan connections, this limited area for clan-based drawing is a series clan (called minimum series clan, MSC). After the MSC is identified, the layout algorithm is applied to the MSC only.

During the incremental drawing, the previous graph drawing and its corresponding parse tree, attributed with bounding boxes, are used to locate the MSC for the next graph and its drawing. The MSC contains all nodes affected by the updates except the added nodes, which are not in the previous drawing. The updates could be multiple node or edge insertions and deletions. For an update, the affected nodes include (a)

nodes added, (b) nodes deleted, (c) nodes connected to deleted nodes, (d) nodes connected to added nodes, (e) nodes connected by added edges, (f) nodes connected by deleted nodes, and (g) user selected nodes.

The following notations are used to denote graph objects in iteration i:

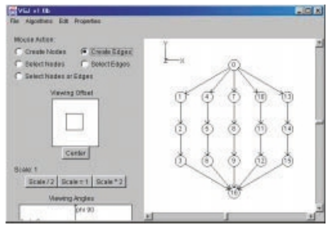
(1) G_i , graph.
(2) E_i , all edges of graph G_i.
(3) V_i, all nodes of graph G_i.
(4) D_i, drawing of graph G_i.
(5) T_i, parse tree of drawing D_i with its bounding box and position attributes.
(6) E_{add}, edges added to drawing D_{i-1}.
(7) E_{del}, edges deleted from drawing D_{i-1}.
(8) N_{add}, nodes added to drawing D_{i-1}.
(9) N_{del}, nodes deleted from drawing D_{i-1}.
(10) $N_{c\text{-}add\text{-}n}$, nodes connected to added nodes N_{add}.
(11) $N_{c\text{-}del\text{-}n}$, nodes connected to deleted nodes N_{del}.
(12) $N_{c\text{-}add\text{-}e}$, nodes connected by added edges E_{add}.
(13) $N_{c\text{-}del\text{-}e}$, nodes connected by deleted edges E_{del}.
(14) N_{sel}, selected nodes.
(15) $N_{affected}$, nodes affected by update. $N_{affected} = N_{add} \cup N_{del} \cup N_{c\text{-}add\text{-}n} \cup N_{c\text{-}del\text{-}n} \cup N_{c\text{-}add\text{-}e} \cup N_{c\text{-}del\text{-}n} \cup N_{sel}$.
(16) P_{i-1}, array of clan tree pointers to leaf nodes of T_{i-1}.

The **msc** and **act** subscripts denote graph objects of the minimum series clan and the subgraph that requires layout computation, respectively.

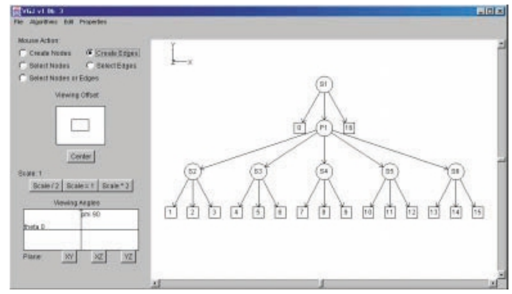
When few changes are made from one iteration of the graph to the next, the new graph can be drawn by:

(1) identifying the MSC, C_{msc}, and its corresponding graph, G_{msc},
(2) adding/deleting nodes and edges from $G_{msc,}$ to form the affected graph, G_{act},
(3) computing the parse tree of G_{act},
(4) determining the layout of G_{act} from the parse tree, and
(5) scaling G_{act} to fit in the space occupied by G_{msc}.

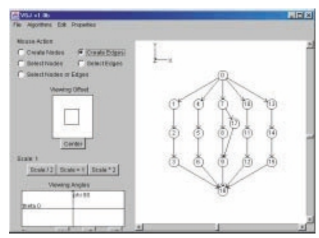
Figure 2 shows a graph drawing, parse tree, and its updated graph. The affected nodes $N_{affected}$ for this update are nodes 7, 9, and 17. From Figure 2 (b) parse tree, the MSC that contains $N_{affected}$ - N_{add} is series clan S4. For Figure 2 (a), the G_{msc} consists of nodes (7, 8, 9) and edges ((7, 8), (8, 9)). After G_{msc} is found, the clan-based layout algorithm will be applied only to sub-graph G_{act} of current graph. The sub-graph G_{act} can be identified as G_{act}.nodes = G_{msc}.nodes $\cup N_{add}$ - N_{del} and G_{act}.edges = G_{msc}.edges $\cup$ E_{add} - $E_{del.}$ In Figure 2 (c), the G_{act} consists of nodes (7, 8, 9, 17) and edges ((7, 8), (8, 9), (7, 17), (19, 9)). After the layout algorithm is applied to G_{act}, the parse tree T_{act} and drawing D_{act} are generated (as shown on Figure 3). The size and position of G_{act}'s drawing D_{act} are attributed in parse tree T_{act}.



(a) Graph Drawing

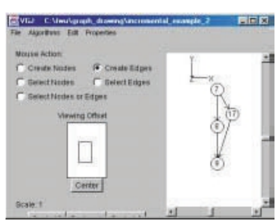


(b) Parse Tree for (a)

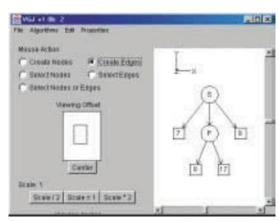


(c) Node 17 and Edges (7, 17) & (17, 9) Added to (a)

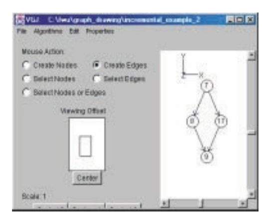
Fig. 2. Graph Drawing, Parse Tree, and Updated Graph



(a) Graph G_{act} of Fig. 2 (c)



(b) Parse Tree of graph D_{act}

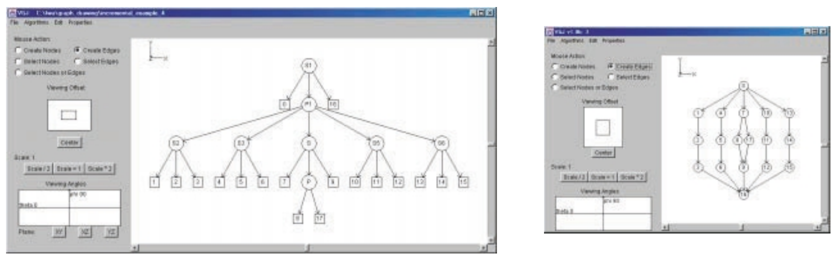


(c) Drawing of Graph G_{act}

Fig. 3. Graph G_{act}, Parse Tree, and Drawing

In order to minimize the number of nodes that must be moved for stable incremental drawing, only the G_{act} is recomputed for current graph G_i, and the drawing D_{act} of G_{act} is sized to be contained in the area used by G_{msc}'s drawing D_{msc}. The size and position of G_{msc}'s drawing D_{msc} are attributed in C_{msc}. If the size of D_{act} is greater than D_{msc}, the D_{act} is scaled down. If the size of D_{act} is smaller than D_{msc}, the D_{act} is positioned in the center of the area used by D_{msc}.

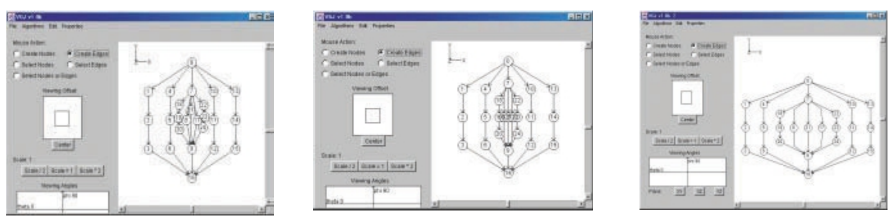
The MSC C_{msc} in T_{i-1} is replaced by graph G_{act}'s parse tree T_{act}. The modified parse tree becomes the current parse tree T_i for current graph G_i. Figure 4 shows the new parse tree T_i and it's drawing D_i of Figure 2 (c)'s incremental update.



(a) Parse Tree from Fig. 2(b) with Series Clan S4 Replaced

(b) Drawing of Figure 2(c)

Fig. 4. Drawing for Figure 2 (c)'s Incremental Update



(a) Updated Graph of Figure 4(b)

(b) Drawing Shows Nodes Overlapped

(c) New Drawing of (b)

Fig. 5. An Updated Graph and Drawings for Fig. 4 (b)

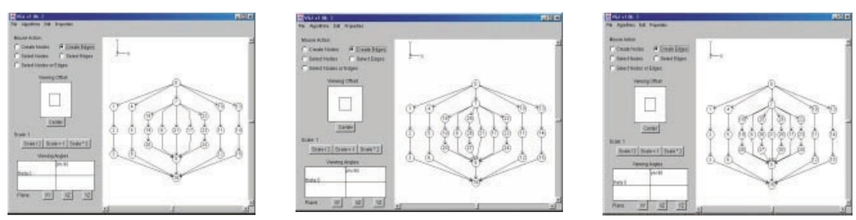
4 Insuring Readability

When D_{act} is scaled down to fit in the area used by D_{msc}, the scaled drawing might be unreadable. Figure 5 (b) shows this problem after an update. Figure 5 (a) is an updated graph of Figure 4 (b)'s drawing. In this updated graph, nodes (18, 19, 20, 21, 22, 23, 24) and edges ((7, 18), (18, 19), (19, 20), (20, 9), (7, 21), (21, 9), (7, 22), (22, 23), (23, 24), (24, 9)) are added. Figure 5 (b) is the drawing for Figure 5 (a). In the Figure 5 (b), the scaled down drawing D_{act} has overlapping nodes that make it a problem for the user to read the drawing. So, a maximum scale limit is needed to ensure readability. Figure 5 (c) shows new drawing of Figure 5 (a). In Figure 5 (c), the scale limit is applied. After the scale limit is reached, the D_{act} is given more space to maintain the readability. Although only the subtree rooted at MSC is replaced, the bounding box and placement attributes of the entire parse tree must be computed. For the example in Figure 5, this has the effect of moving nodes (1, 2, 3, 4, 5, and 6) to the left and nodes (10, 11, 12, 13, 14, and 15) to the right.

5 Locality of Incremental Drawing

Successive modifications to graphs often occur within a small geometric or logical neighborhood [10]. In order to build more stable drawing with fewer computations, the algorithms should take advantage of this locality property. When adjusting the previous clan tree T_{i-1} and the previous drawing D_{i-1} to make extra space for D_{act}, if the space created is the exact space needed by D_{act}, then any nodes added to the current G_{act} might cause T_i and D_i to be adjusted again. If the graph is expected to grow, the scale factor for G_{act} should be increased.

Figure 6 shows an example of extra space created for future updates. Figures 6 (b) is successive drawings of Figure 6 (a) and Figure 6 (c) is successive drawing of Figure (b). In Figures 6 (b) and 6 (c), nodes (1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15) are not re-positioned because the possible extra space needed by successive drawings has been generated during Figure 6 (a) drawing layout computation.



(a) New Drawing of Fig. 5 (b) | (b) Successive Drawing of (a) | (c) Successive Drawing of (b)

Fig. 6. New Update Drawing of Figure 5 (c) with Nodes and Edges Added

6 Reposition Nodes Not in the Minimum Series Clan

If more space is needed, what nodes need be re-positioned or re-sized? Let $P_{msc\text{-}root}$ be the path from C_{msc} to root in previous parse tree T_{i-1}, and L_{adjust} be extra length and W_{adjust} be extra width needed by D_{act}. To make extra space, for each clan C_{path} along the path $P_{msc\text{-}root}$ needs to add W_{adjust} to C_{path}'s width and L_{adjust} to C_{path}'s length. Also, along the path $P_{msc\text{-}root}$,

1. all left siblings of C_{path} whose parent is a parallel clan need to be shifted left with $W_{adjust}/2$ distance,
2. all right siblings of C_{path} whose parent is a parallel clan need to be shifted right with $W_{adjust}/2$ distance, and
3. all right siblings of C_{path} whose parent is a series clan need to be shifted down with L_{adjust} distance.

Figure 7 is a parse tree with C_{msc} identified. The path $P_{msc\text{-}root}$ includes clans C_{msc}, P7, S5, P2, and S1. In this parse tree, if more space is needed :

1. node 64 and clan S4's nodes need to be shifted to the left,
2. clan S10's nodes and clan S6's nodes need to be shifted to the right, and
3. and node 65, clan P8's nodes, and P3's nodes need to be shifted down.

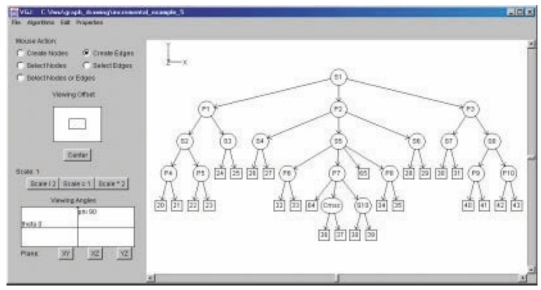
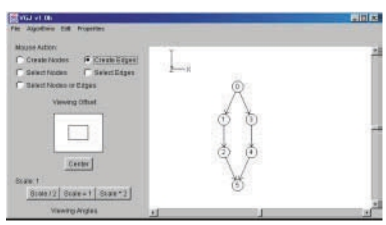


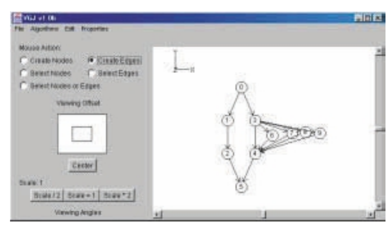
Fig. 7. Parse Tree with C_{msc} identified

7 Resizing Affected Clans

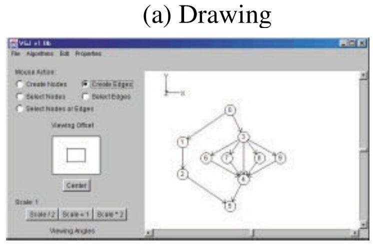
Is it necessary to resize all clans along the path $P_{msc\text{-}root}$? For a parallel clan, its length is the max value of all children's lengths. For a series clan, its width is the max value of all children's widths. For each clan C_{path} along the path $P_{msc\text{-}root}$, if both the C_{path} 's width and length do not exceed parent's max values, it is not necessary to resize parent.



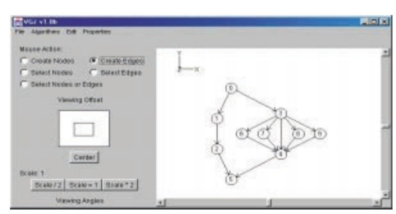
(a) Drawing



(b) Updated Graph of Drawing (a)



(c) Drawing for (b)

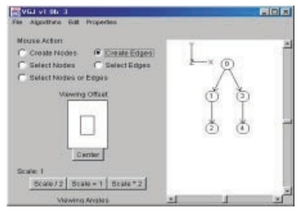
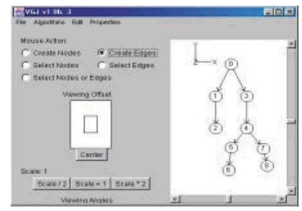


(d) New Drawing of (b)

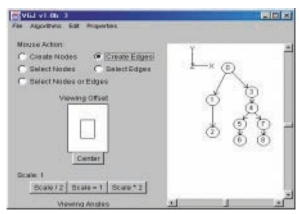
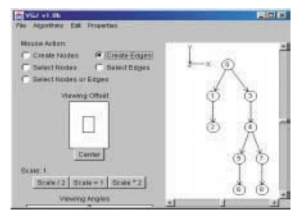
Fig. 8. Updated Graph and Drawings

In some situations, it is not necessary to re-position siblings along the path $P_{msc\text{-}root}$ or resize clans which contains C_{msc} when the drawing D_i requires extra space. In Figure 8 (b), nodes (6, 7, 8, 9) and edges ((3, 6), (6, 4), (3, 7), (7, 4), (3, 8), (8, 4), (3, 9), (9, 4)) are added to Figure 8 (a) drawing. Figure 8 (c) is the drawing for Figure 8 (b). In Figure 8 (c), nodes 1 and 2 are shifted to the left to make some space for updates. However, the shift is not necessary. Since there are no nodes at the right hand side of nodes 3 and 4, it would be more reasonable for updates to grow toward

the right without shifting nodes 1 and 2. Figure 8 (d) shows a different drawing of Figure 8 (b). In Figure 8 (d), nodes 1 and 2 are not moved.

(a) Drawing (b) Updated Graph of Drawing (a)

(c) Drawing for (b) (d) New Drawing of (b)

Fig. 9. Updated Graph and Drawings

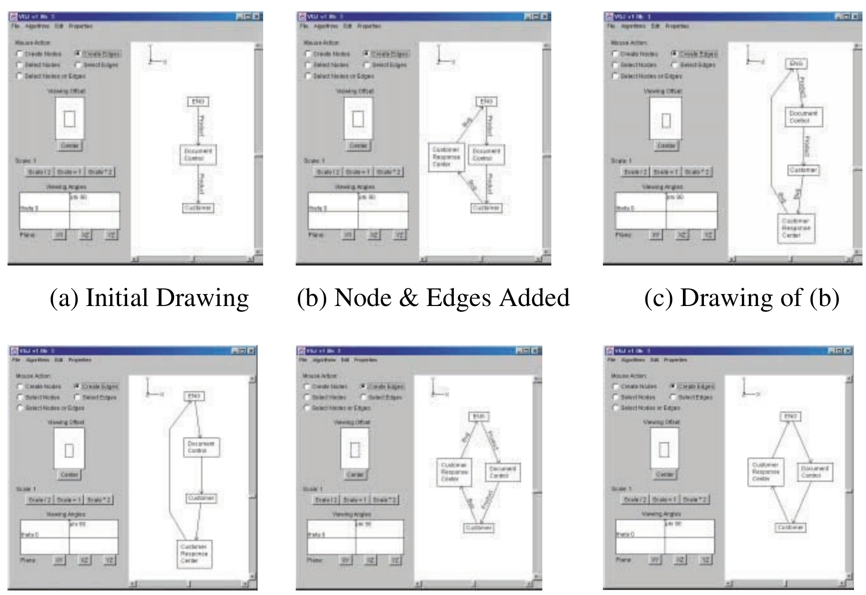
Figure 9 shows a case when re-sizing is not necessary. In Figure 9 (b), nodes are appended to the node 4 of Figure 9 (a). Figure 9 (c) is the drawing for Figure 9 (b). In Figure 9 (c), the sub-graph containing nodes (3, 4, 5, 6, 7, and 8) is scaled even when there are no nodes below them. In this case, the graph should be able to grow downward freely without affecting the drawing stability. Figure 9 (d) shows a different drawing of Figure 9 (b). The path $P_{msc\text{-}root}$ can be used to identify the cases where re-sizing / re-positioning is not necessary:

1. along the path $P_{msc\text{-}root}$, if there are no left siblings for a clan whose parent is a parallel clan, the drawing D_i can grow toward left without shifting other clans to the right,
2. along the path $P_{msc\text{-}root}$, if there are no right siblings for a clan whose parent is a parallel clan, the drawing D_i can grow toward right without shifting other clans to the left, and
3. along the path $P_{msc\text{-}root}$, if there are no right siblings for a clan whose parent is a series clan, the drawing D_i can grow downward without scaling down length.

8 Incremental Drawing of Cyclic Graphs

Using the depth first search to find an edge to be reversed for the cyclic graph may not be suitable for some applications [12]. Figure 10 shows a problem in incremental drawing when only one edge is drawn upward for each cycle of cyclic graphs. Figure 10 (a) shows a path of a company's system code release. In Figure 10 (b), a new path for bugs report is added. Figure 10 (c) is the drawing of Figure 10 (b). The Figure 10 (c) does not represent the two paths in the expected way. If edge labels are removed (as Figure 10 (d)), it will be even more difficult for the user to visualize the concept of

two different paths. In order to improve this situation, the visual input graph nodes' position information is used. During the graph layout computation, if upward edges contributed to a cycle are found, those edges will be reversed for layout computation and then be changed back to upward after layout computations done. Figure 10 (e) shows new drawings of Figure 10 (b). As shown in the Figure 10 (f), even when the edges are not labeled, the drawing still represents the concept of two paths.

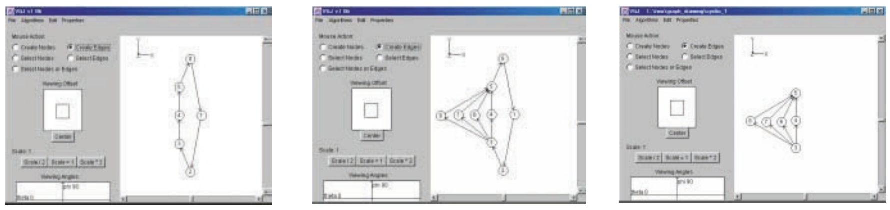


(a) Initial Drawing (b) Node & Edges Added (c) Drawing of (b)

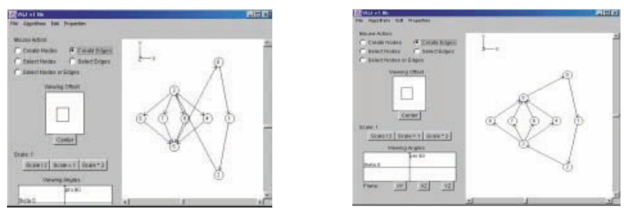
(d) Drawing of (c) with Labels Removed (e) Drawing of (b) (f) Drawing of (b) with Label Removed

Fig. 10. Updated Graph and Drawings of Incremental Cyclic Drawing

By using clan-based drawing for incremental drawing, the layout computation only applies to the affected sub-graph G_{act} instead of the entire graph. Sometimes when the sub-graph is extracted from the entire graph, its original role in the entire graph might be missing. The Figure 11 (b) is an update of Figure 11 (a), and Figure 11 (c) is Figure 11 (b)'s G_{act} which requires layout computation. The G_{act} is not a cyclic graph, but G_{act} is part of the cycles of Figure 11 (b) cyclic graph originally. Since G_{act} is not a cyclic graph, no edges are reversed during the layout computation. Figure 11 (d) shows the incorrect updated drawing of Figure 11 (b). To insure proper edge direction, upward edges need be checked to see if those edges are part of cycles of original graph. If an upward edge contributes to a cycle, this upward edge will be reversed during the computation and reversed again after the computation. Figure 11 (e) shows the correct drawing of Figure 11 (b).



(a) Initial Drawing (b) Nodes & Edges Added (c) Sub-graph of (b)



(d) Incorrect Drawing of (b) (e) Correct Drawing of (b)

Fig. 11. Drawing and Graph with Upward Edges Added

9 Incremental Drawing Examples

Figure 12 shows a series of incremental drawings created by the clan-based drawing algorithm. The Figures 12 (b), (e), and (f), show that some upward paths were added. Those upward paths are part of cycles. After nodes (11, 12, 13) and edges ((4, 11), (11, 12), (12, 13), (13, 6)) are added to Figure 12 (f), the length of new drawing, Figure 12 (g), is not changed because the minimum affected area still has enough space for updates. Figures 13 (n) is the new drawing of 12 (m) after upward edges ((7, 18), (7, 20)) are added. The edges ((7, 18), (7, 20)) are not part of cycles, so node 7 is moved to above nodes 18 and 20.

10 Conclusion

Clan-based graph drawing using a parse tree improves the incremental drawing stability for directed graph drawings in several areas:

(1) The attributed parse tree provides an easy way to layout graphs with nodes of different sizes. A node can be spanned over more than one level in drawing if necessary. During the incremental updates, if the change is only to enlarge nodes, the new updates might be done very easily without re-positioning other nodes.

(2) The utilization of parse trees allows the incremental drawing to be stable without sacrificing aesthetic criteria and speed. By using clan-based drawing, the incremental drawing can be done very efficiently because the changed and recomputed area can be localized with the provided parse tree.

(3) No extra constraints are needed to maintain incremental drawing stability. For clan-based drawing, the node position constraints are embedded in the parse tree.
(4) By using the parse tree, it is easy to determine whether the modifications are in the interior or the exterior boundary area of a drawing. If the changes are made to exterior area, the updates can grow freely toward the open area without re-positioning other nodes.
(5) By using the parse tree to locate the minimum affected area of updates, the locality issue can be considered for the future stable updates.

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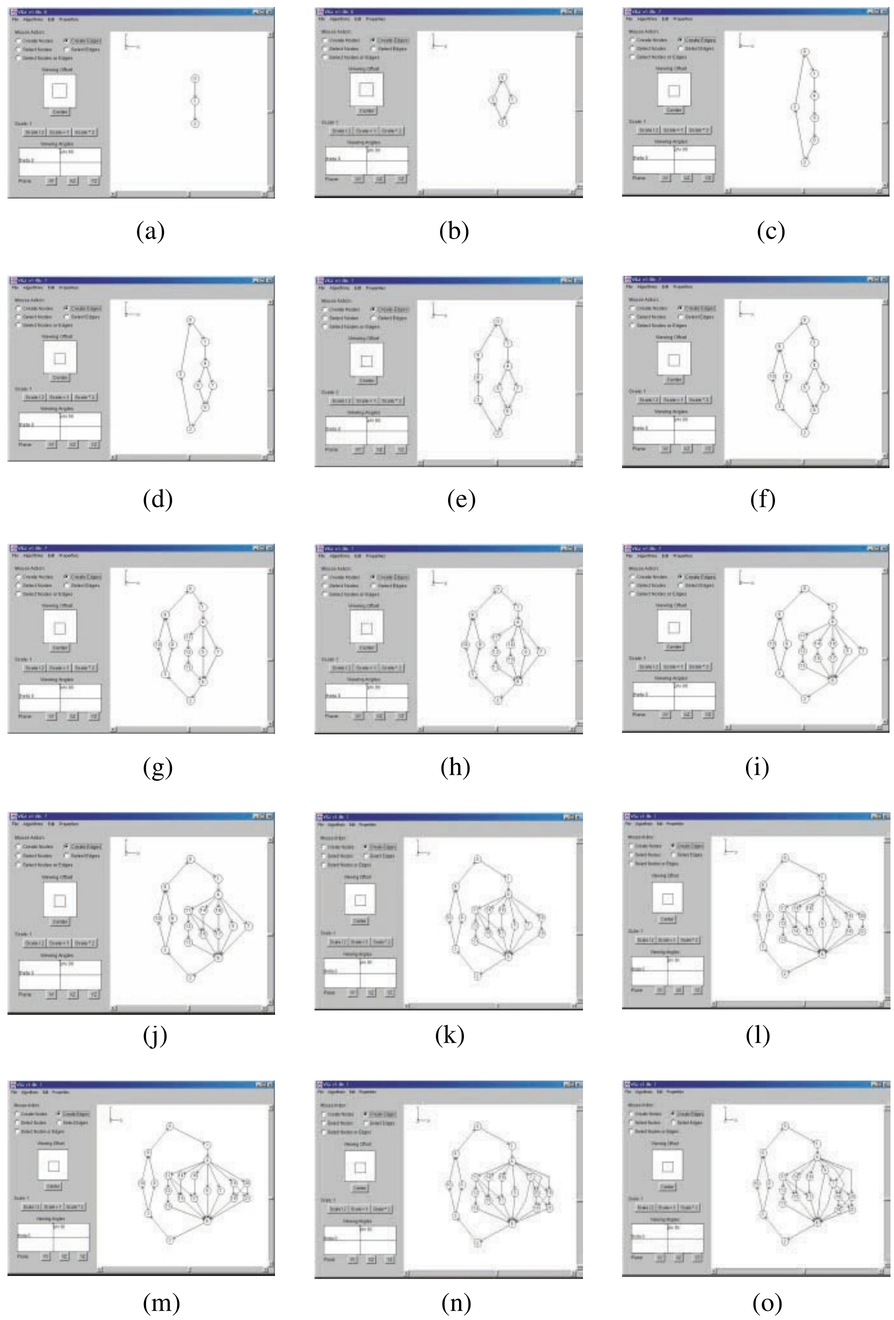

Fig. 12. Clan-Based Incremental Drawings

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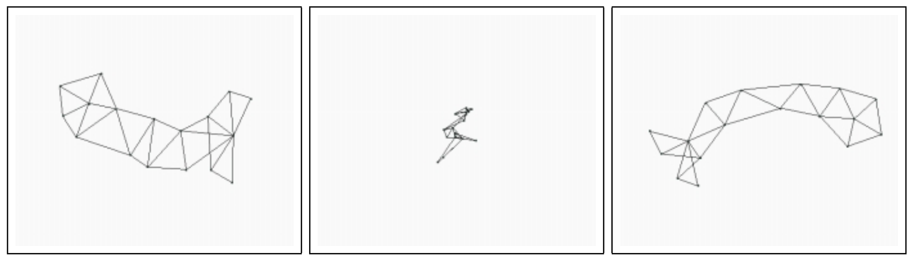

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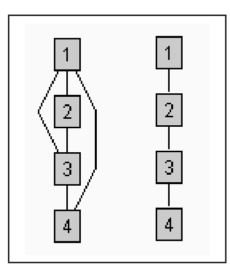

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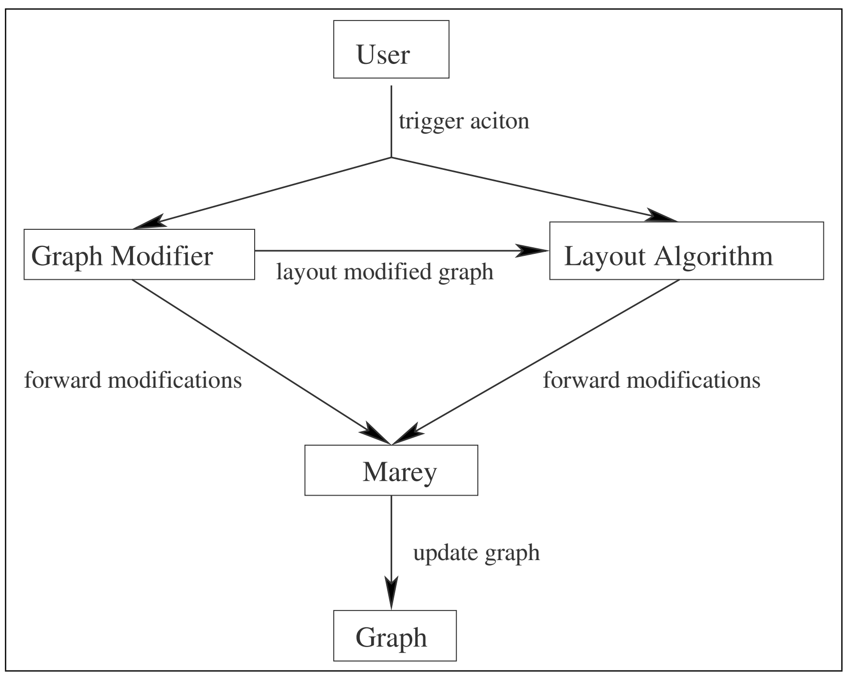
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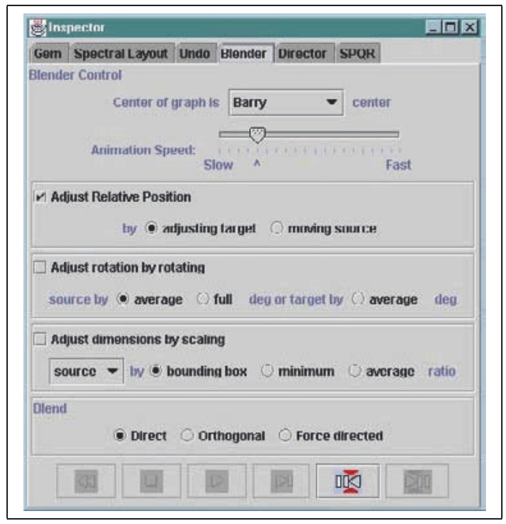

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r c t r r t c t r t t r t
l st st t r t t t r t t s l
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r r r t r c s t c t sc l r
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r t l ts s r l
l r t s r l l s st r c st t s t r
t t r st s l r t lt r t l t s ts c
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ll l s s t c l s c s s t t l
l c t r t ct r s t s s ct tr t c t r t t
r c ss c rs t s s l ss l t c rt r ll l s r
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http://www.cs.newcastle.edu.au/~friedric/gd00
t t t s r l l l s l r r s t s rt t
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rst l s s t t t t rc r ct l ts
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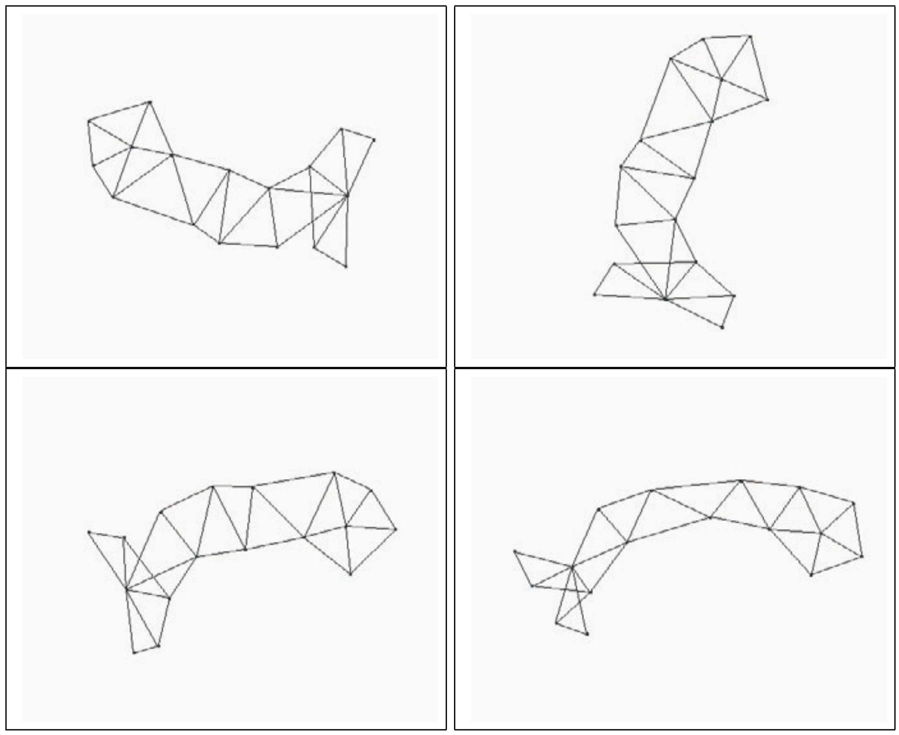

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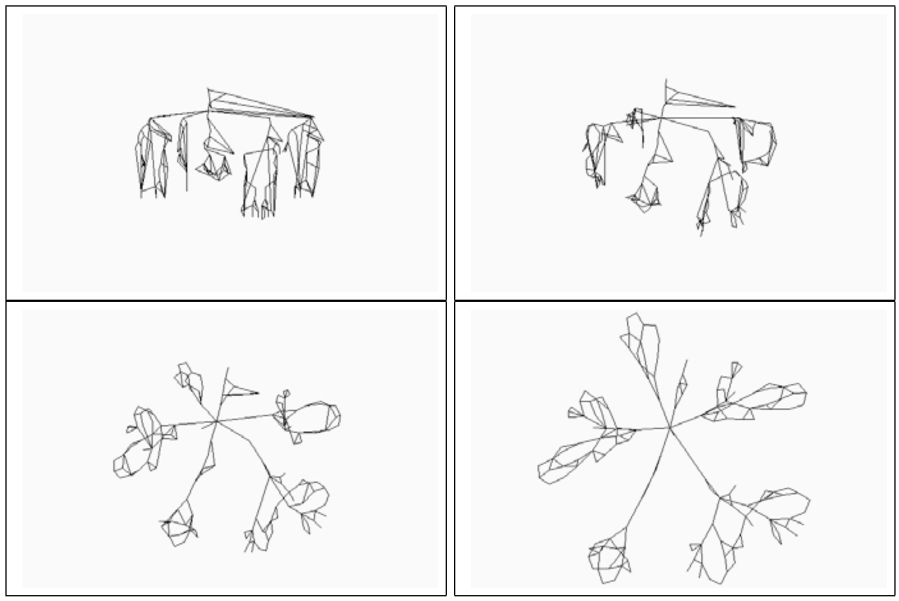

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3. . r ul . r r l r r s r s r ss r r s. *ss* *s* 2) 3 2 .
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. r k u l u. s l u l r r u r r s. *s* .
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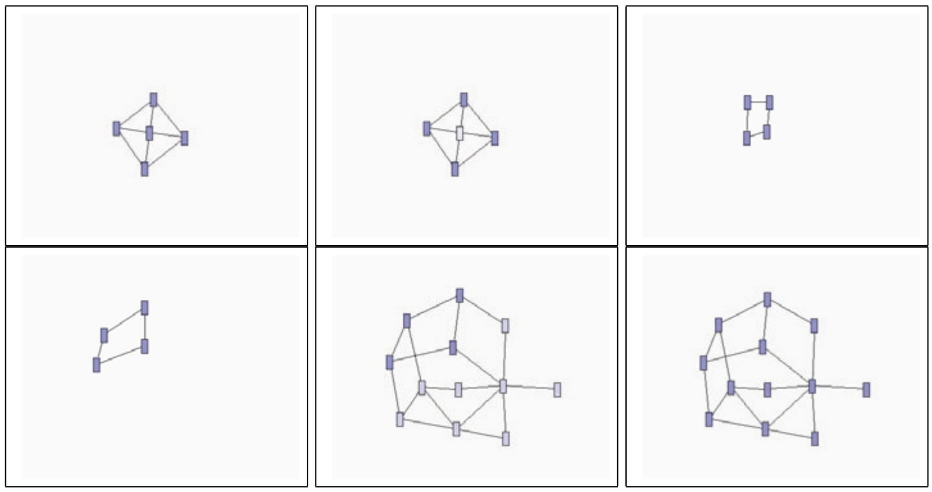

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2. r . r. *s* *s* *s*
s. s s rs s l us r l) .

Ulrik.Brandes@uni-konstanz.de
scott@cwi.nl
north@research.att.com

http://www.graphdrawing.org/data/format/

http://www.cs.virginia.edu/~gd2000/

2.

http://www.graphdrawing.org/data/format/.

http://www.gupro.de/GXL/

http://www.cs.toronto.edu/~simsuz/wosef/

r r rg lrik r s
i l imsolt r s it r

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brandenb@fmi.uni-passau.de
t¨t o t
o t ,
l l o u t u l
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po t d t t u l p
o t t, ld o ju t o t t 2 p po u
ll u , . pu po o t o t t to o to d
ll t u t t t o t t p d t olo , ,
, , ,2 .

t s riptio s o t o r t gori s or t 2 o t st r il l i t
orl i ( ) . ig t s p r t s missio s r r i o-
t i i g 33 i r t gr p r i gs o li mo str tio . or o r t r
r o r spo t o s mo il s o sp l st rs. i rs or t t gori s
r s l t t o t st org i rs o rks. o fli ts o i t r st r
oi o o or sis. i i g tri s r s ri lo .

gr p gi or t gor s g r t it iml r r sl r s
l r. l r is tool or st ti l sis o so t r .
t s pports is li tio o so t r it gr p s ro si g ross r r i g
s ll s omp t tio o m tri s.

is p rti l r gr p is r l orl mpl is li s t l ss r l-
tio s ips i t l r its l . gr p o sists o l rg m r o
i i i l ompo ts o t i s s r l ig gr o s. ll g or
t is p rti l r o t st tr s ot to is li i t r sti g gr p prop rti s
(t r pro l r t) t to i ompr si l l o t or
r t r l rg gr p .

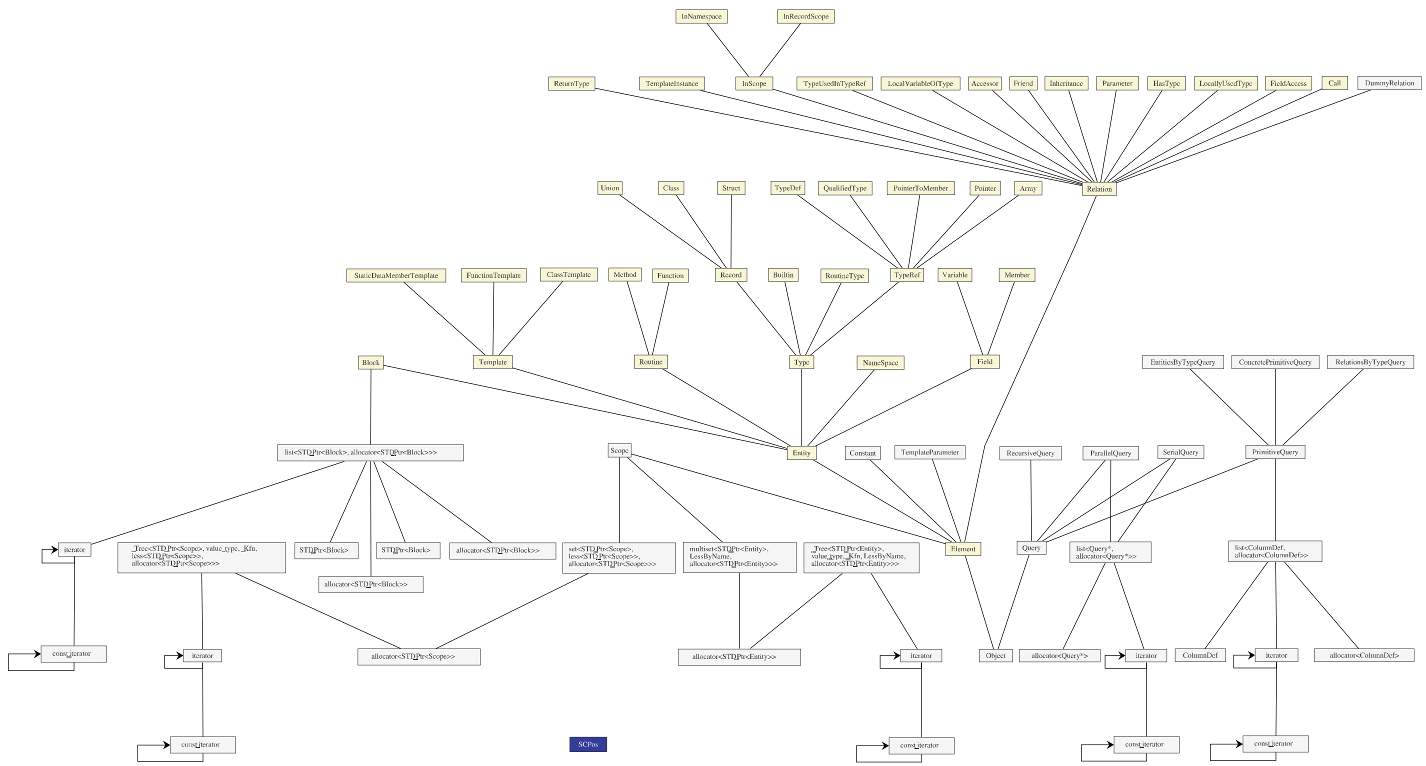
mo g t s mitt tri s t i r ikol . ikolo rom t i r-
sit o im ri k g t st l sis ispl o t rl i g str t r .

○

is r i gs r m it - s s st m lop t r p r i g r s r gro p t t i rsit o im ri k
rst st p s to s p r t t 26 o t ompo ts o t gr p .
ompo t s r s p r t l si g t - s lgorit ms or l ri g rossi g mi imi tio . ll t r i gs mi im m m r o mm o s or t l ri g lgorit m (s j t to t gi i p t p r m - t rs) mi im m m r o g rossi gs or t r s lti g l ri g. l r i gs o t ig ompo ts r os mo g lt r ti sol tio s m it i r t i p t p r m t rs to t l ri g lgorit m. t s tri to k p t im sio s o t r i g it i r so l o s to i s g rossi gs s possi l . l iti g s to t t positio o t o s to ti t l ls.
t t o igg st ompo ts t rro s o t g s r r mo to i l r r r i g. ll t g s it o t rro s poi t o r s.
o s o ompo t 2 r olor i t r olors it llo (gr i t pri t) l (l k i t pri t). llo o is o t g to t l o it l l " os . s g s r r mo to i tt r r i g. ll t s g s llo so r o t l o " os s sti tio o . ig r s o s t is ompo t.

• •• • • • • • • • •

t l sis o so i l t orks gr p r i g is i r si gl r og i s tool to ti l s pport is l plor tio omm i tio o i gs. gr p s gi or t gor r ss o o t m jor ll g s i t is r m l t omp riso o t orks o simil r t p .
t or t gor r pro i r . rl i - p t si ss o s lt t. it i si l orpor tio t o t ms orki g i simil r proj ts r sk o r q t t o l i o t t it ot r m m rs o t ir t m it t ir li ts it om i p rts t r l to t ir t m. t t s o sists o t o ir t gr p s it t r t p s o rti s (t m m m rs li ts t r l p rts) i t g r g ig ts r gi g rom to 4 (i i ti g q rt rl mo t l i kl kl o t t r sp ti l).
s i m gr p r i g o t sts or t i i g tr s s mit- t l imir t g lj r j r r rom t i rsit o j lj lo i . ig r 2 s o s t ir ir l r l o ts o t t o t orks r t m m m rs li ts p rts r lo t o t i r mi l o t r ir l r sp ti l . ss ti ll t ir l r or ri g is t rmi t m t l o r- l p o p ir is rt ig or oo s m l ppl i g risti or t r - li g l sp rso ro l m to ompl t gr p it g ig ts t t q ti ig or oo simil rit . g ig ts r pi t i r t gr s l s.
s im g s l rl s o o ot t ms i i i to li t/ p rt i t r- gro p orki g i t r ll . o r t t ms lso i r q it isi l i ot t ir i t r l t r l omm i tio . r i gs t s r il l omp riso i t rpr t tio o t t s t ms org i t ir o t ts.



InNamespace
InRecordScope
ReturnType
TemplateInstance
InScope
TypeUsedInTypeRef
LocalVariableOfType
Accessor
Friend
Inheritance
Parameter
HasType
LocallyUsedType
FieldAccess
Call
DummyRelation
Relation
Union
Class
Struct
TypeDef
QualifiedType
PointerToMember
Pointer
Array
StaticDataMemberTemplate
FunctionTemplate
ClassTemplate
Method
Function
Record
Builtin
RoutineType
TypeRef
Variable
Member
Block
Template
Routine
Type
NameSpace
Field
EntitiesByTypeQuery
ConcretePrimitiveQuery
RelationsByTypeQuery
Scope
Entity
Constant
TemplateParameter
RecursiveQuery
ParallelQuery
SerialQuery
PrimitiveQuery
Element
Query
Object
ColumnDef
iterator
const_iterator
SCPos

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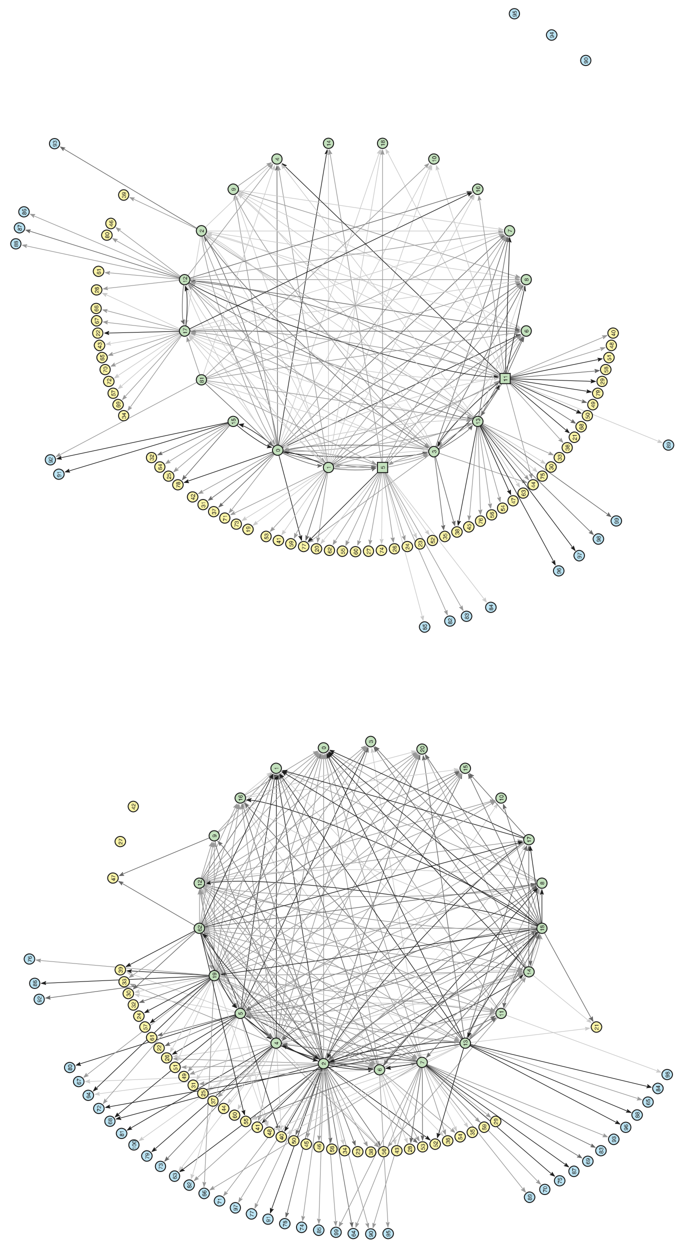


• •• • • • t o t o o l olo)

•.. • ••••••• •

t or t t gor gr p r pro i o rlis o t i- rsit o i sot o o st i i polis. gr p s ri s r l i - s t -mo l lt o g ll t l ls r g to s to prot t t i t ll t l i stm t.

t mo l is t olog -i p t st t m t o t ki s o t to r m m r t s m g m t s st m. or t is o t st sim- pli t r i g to i l j st titi s ttri t s i r r l tio s ips. r p i ll tit pp rs s o (r t gl) ttri t pp rs s t t it i tit s o r l tio s ip pp rs s li t t o (ot ss ril isti t) titi s. q lit o t gr p sig i tl imp ts t sig pro ss. poorl r gr p m s r l mo l.

ort t l t r r o tri s or gr p it is k pt or t r omp titio s.

•.. • ••••••• •

t gor is t r or rtisti t gor . r is o pli it ll g gr p t r m ork to om i rts gr p s. tri s to t gor r i i i to t r s - t gori s

• •• m rom rl s i rsit r g took ompositio o iro o rt t p i ti g i to gr p p tti g rti s t i t rs tio poi ts i g som rti s g s.

o its i ompo ts s r s p r t l si g mo i spri g lgorit m. o rt r m l iti g (pt o i g olors) s o .

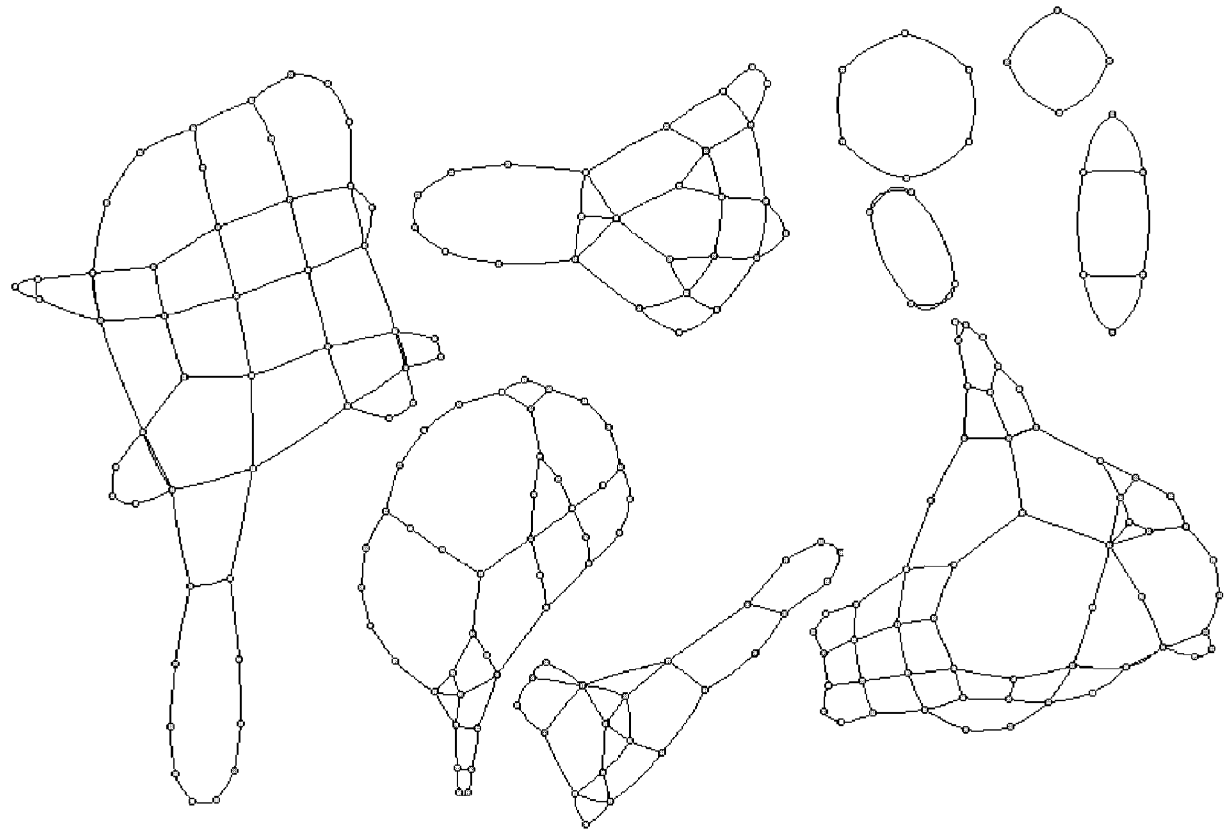
• •• risti . l j r i l . oo ri t p . o o ro s mitt st r o-gr p i ologr m o t gr p orr spo i g to t i rpi ski p r mi o or r 7 s ig. 4 or 2- r i g o t ir 3- mo l.

i rpi ski p r mi is l ssi r t l . r s tr itio ll t im g is it rti s g s t m t irs r t l gr p it o sp i m i g. s gr p m r to m t r pr - s t tio o t gr p pro i g i t r sti g ti l r s lts.

i rpi ski p r mi gr p is r t r rsi pro r p r m- tri o t or r o t r rsio . s i t 2- s t it r tio r p r mi is i i i to o gr t sm ll r p r mi s it t tr l p - r mi r mo . i rpi ski p r mi o or r • t m r o rti s is •. $\dot{-}$ +2 t m r o g s is •. 6(• −4)+ 2 3 4•. i t ir mpl t i rpi ski p r mi o or r 7 s 94 rti s 49 2 g s.

i t p r m t r • t j m tri o gr p i orr spo s to t i rpi ski p r mi o or r • s g r t . gr p is t r si g t GRIP s st m 9 it o t mo i tio to t r s lti g r i g.

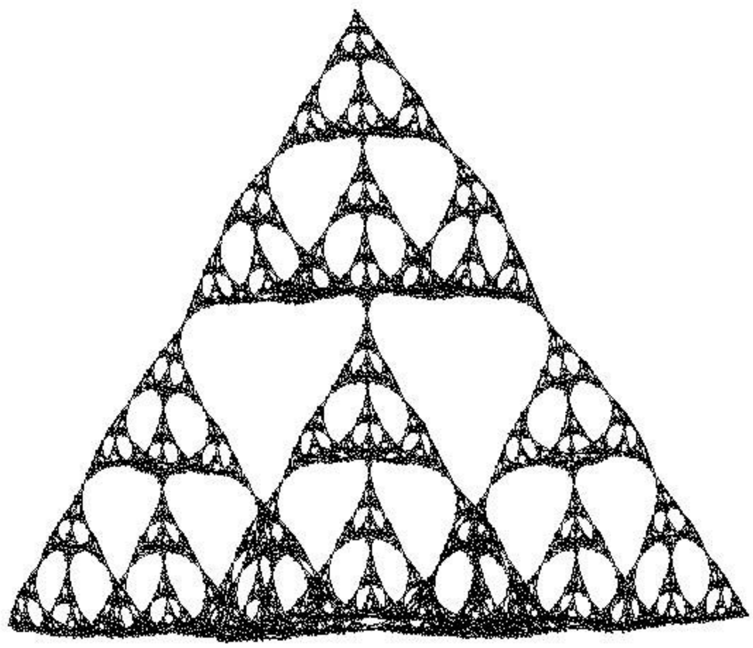
r i g m t o o t GRIP s st m s s m lti- im sio l lgorit m



•••••• d ' o o o' p t o l olo)

. d u t l.

or r i g l rg gr p s . lgorit m t k s t g o t s mm tri t r o t i rpi ski p r mi to q i kl pro l r i g. ltog t r t g r tio o t gr p t pr pro ssi g l 3- r i g took 22 s o s o ti m it 2 o .



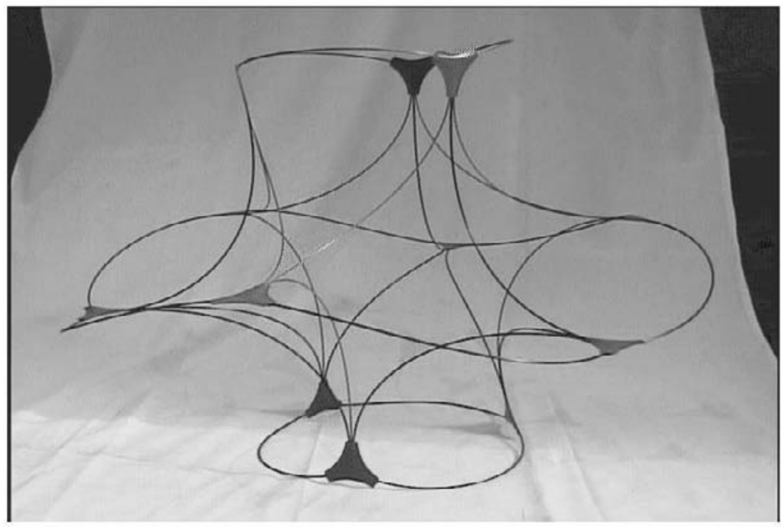
• •• • • • 2 d o t olo o t p k p d o o d .

ologr m o t r i g s r t si g t st r o-gr p i im g m t o . ologr m pt r s t 3- t r o t r i g to pro t simil r to t t o rot tio o t origi l 3 mo l i p . ologr m its l s pr s t t t o r is i it t t i rsit o ri o .

• •• s o rks s rpris or t p rti ip ts o t s mposi m. o sp l st r i t ir 2 g. sp l st r o sists o 72 l sti ro s 24 pl sti i t rs tio s. p rti ip ts r o r g to o str t t r im sio l gr p s. r r s rp tim r stri tio s t r r ks o ig t s ssio . o r t ms s mitt t ir ss m li s i r st ti ll i 3 gr p s.

i r is " os o ö l rom t i rsit o ll rm ikol . ikolo rom t i rsit o im ri k r l . ig r s o s t p oto. is is t l rg st it gr p i is ot m im l pl r r g l r.

p o t t po t



“ o ”

org i tio o t is r s gr p r i g omp titio s i t rom its i iti tor o rks to r r rg. o ro t it t t r t sks o oll ti g ll g gr p s r isi g mo rom spo sors ll r ki g t tri s t rst o s t r st.

s i t p st rs t gr p r i g omp titio s i ti t gor . r l tim s t post ll g gr p s s r to i iti t r s r or gi p s i p rti l r ir tio . i i g tri s r r r i s r l p p rs. is r t t r t k rom r l li pro l ms. si g ll- o ppro s t i rs m t rl i g m i g o t t isi l .

ort t l t m r o o tri tio s to t gr p r i g omp titi- o s s r lo poi t. ll tr to ttr t mor m m rs i si o tsi t gr p r i g omm it to o tri t to t gr p r i g omp titio .

po sors ip or t is o t st s pro i - s r lor m rk iml r r sl r ors gs tr m lm om - r o t r rk l .

. d u t l.

• ••••••••

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t ' , p
, l , pt . p .
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o k, . .